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TWO MENTAL CALCULATION SYSTEMS: A CASE STUDY OF SEVERE ACALCULIA WITH PRESERVED APPROXIMATION

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Abstract—We report the case of an aphasic and acalculic patient with selective preservation of approximation abilities. The patient's deficit was so severe that he judged $2+2=5$ to be correct, illustrating a radical impairment in exact calculation. However, he easily rejected grossly false additions such as $2+2=9$, therefore demonstrating a preserved knowledge of the approximate result. The dissociation between impaired exact processing and preserved approximation was identified in several numerical tasks: solving and verifying arithmetical operations, number reading, short-term memory, number comparison, parity judgement, and number knowledge. We suggest the existence of two distinct number-processing routes in the normal subject. One route permits exact number representation, memory and calculation using symbolic notation. The other route allows for approximate computations using an analog representation of quantities.

INTRODUCTION

IN THE RECENT years, detailed case studies have significantly increased our comprehension of acalculia. On the joint basis of studies with normal subjects and with patients, models of the normal architecture for number processing have been proposed [6, 15, 28], the lesioning of which predicts the typology of impairments found in acalculic patients. According to MCCLOSKEY's influential model [27], separate components of number comprehension and number production interface with an abstract semantic representation, on which the various calculation routines operate. The various patterns of number production deficits have been explained by different types and loci of damage within the production module [27]. Likewise, various calculation deficits have been described, resulting either from a failure to access stored tables of arithmetical facts [29, 46], or to sequence correctly the successive elementary operations [7], or even to process correctly the operation sign [17].

If transcoding and calculation are well studied, the nature of the semantic representations accessed in number comprehension remains a debated issue. DELOCHE and SERON [15] attempt to model number processing without even resorting to an intermediate level of semantic representation. On the contrary, MCCLOSKEY [7, 27, 28] sustains that numbers are necessarily translated into an amodal semantic representation. However, in the current formulation of the model, this semantic representation is so similar to Arabic notation that labelling it 'semantic' seems somewhat artificial. Finally for CAMPBELL and CLARK [6], numbers evoke an 'encoding complex' in which multiple representational codes are

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activated, including 'semantic', 'analogue' and 'imaginal' ones. However, in the absence of a more precise definition for these codes, this proposal is difficult to evaluate or to falsify. In brief, none of these models have provided satisfactory accounts of how we access the *meaning* of a number in a given context (e.g. Is this a reasonable price for this article? Does this operation look correct? Which of these two quantities is the smallest?).

The numerical quantity that a number represents is an important aspect of the semantic information to be recovered during comprehension. Experiments with normal subjects have suggested the existence of an analogical representation for numerical quantities [11, 12, 37]. After translation from, say, the Arabic notation, numbers would be represented mentally in the same way as physical magnitudes like size or weight (analogical encoding). Data from the number comparison task support this hypothesis [32, 33]. Subjects behave identically whether they have to choose the larger of two objects [34] or the larger of two numbers [5]. In both cases, comparison times decrease with increasing distance between the stimuli, suggesting that in some sense numerical distance is directly analogous to physical distance. This distance effect extends to 2-digit numbers. For instance it takes longer to compare 49 to 55 than to compare 41 to 55, despite the fact that in both cases the response 'smaller' is determined by the decades digits [11, 12, 23]. The numerical comparison results can be explained by supposing that each number activates a fuzzy region on a mental number line [37]. Numerically close numbers such as 49 and 55 would activate overlapping regions, hence they would take longer to compare. The hypothesized mapping of numbers onto the number line has been studied by various psychophysical techniques, and is assumed to obey Fechner's Law: for equal numerical distance, the larger the two numbers, the closer they are on the internal representation [5, 11, 25].

Several animal species, including rats and pigeons, are capable of processing and comparing approximate numerical quantities [10]. Human infants, well before they acquire language, can discriminate small numerosities [42], match them across the auditory and visual modalities [43] or select the larger of two small numerosities [8, 40]. GALLISTEL and GELMAN [19] postulate that approximate numerical quantities are handled by a preverbal system, available to animals and infants, and which, in humans only, serves as a foundation for the acquisition of language-based counting and calculation abilities. In adulthood, this representation of approximate quantities would keep a central role in conveying the meaning of numbers processed in Arabic or verbal notation.

We report here the case study of a severely aphasic and acalculic patient, N.A.U., who lost all precise knowledge of numbers and arithmetical operations, and could only operate with approximate numerical quantities. For instance N.A.U. could not rapidly solve an addition like $2+2$, nor reject $2+2=5$ as false, but he recognized $2+2=9$ as incorrect. A similar dissociation was found for several cognitive functions, including short-term memory which was good for approximate quantities but poor for exact digit identities. We suggest that N.A.U. lost most of his acquired language-based number processing abilities, but that his analog representation of approximate quantities was intact.

CASE REPORT

N.A.U., a 41-year-old executive salesman, suffered a severe head trauma and underwent surgery for a left extradural hematoma. Post-surgery, he showed aphasia, right hemiparesis and right hemianopia. CT-scan disclosed a large temporo-parieto-occipital hypodensity (Fig. 1). Testing was carried out 3 years after the trauma. At that time, a French version of

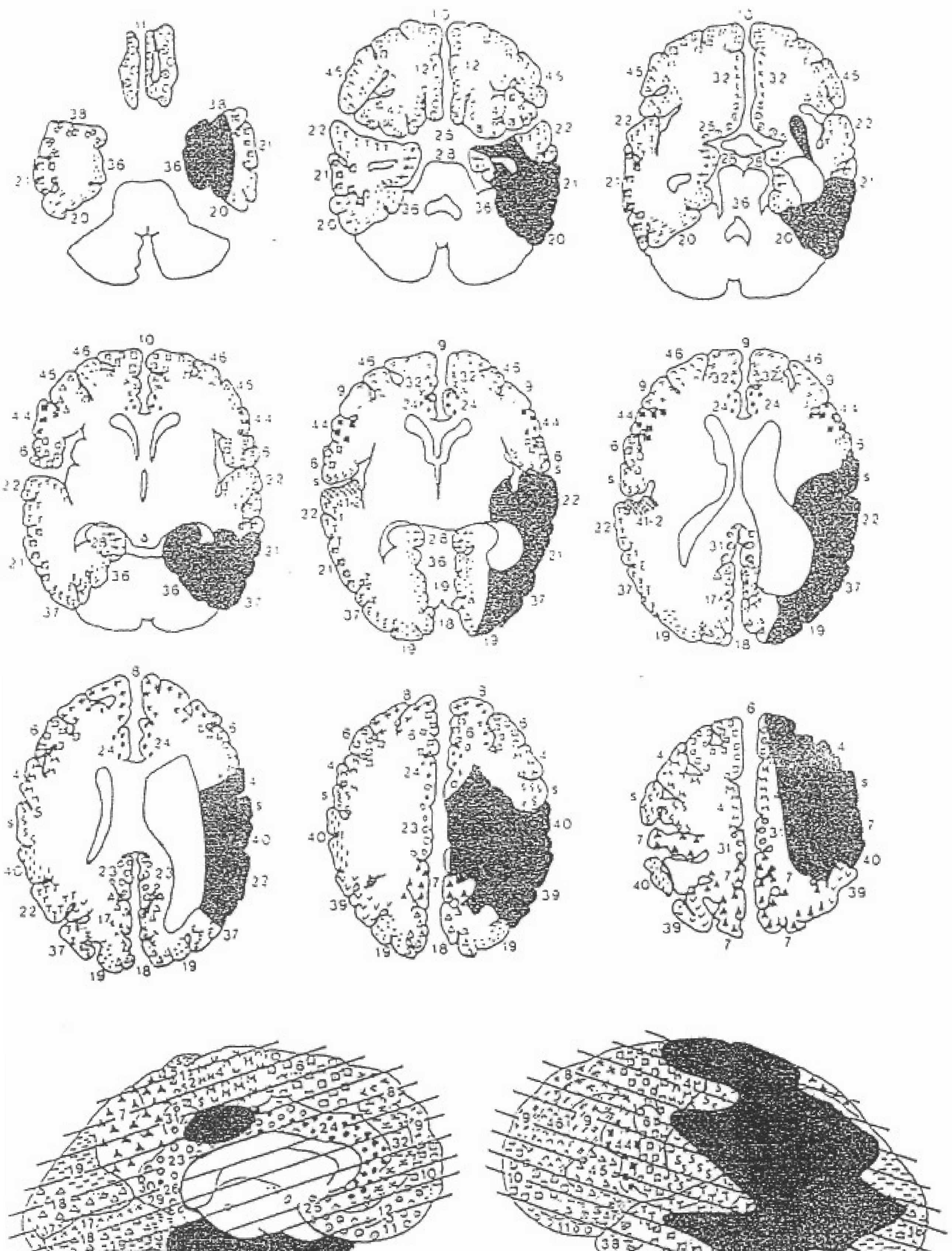

Fig. 1. Reconstruction of N.A.U.'s brain on the basis of CT scan using templates from DAMASIO and DAMASIO [9]. The lesioned left-hemisphere is represented on the right side of transversal sections.

the Boston Diagnostic Aphasia Examination (BDAE) [21, 26] revealed a moderate impairment of oral language, affecting both comprehension and production. N.A.U. was much more severely impaired with written language, failing to read or write most of the stimuli. His digit span was also severely reduced (forward: 3; backward: 2).

A closer analysis of the BDAE data revealed that in several subtests, performance was better with number words than with other words. For instance in the word reading subtest, the only word that N.A.U. could read was 'dix-huit' (18). In the word picture-matching subtest, he succeeded only in matching 'dix-huit' (18) and 'sept cent trente' (730) with their Arabic counterpart. In dictation, he was approximately correct with numbers, writing 7, 43, 198 and 1985 in response to the stimuli 7, 42, 193 and 1865; by contrast he completely failed in dictation of letters and words. These preliminary results therefore suggested a relative sparing of number-related abilities, which is further documented below.

CLINICAL ASSESSMENT OF READING

Several further tests were performed to clarify N.A.U.'s abilities in reading words and numbers. N.A.U. correctly read only 2 out of 12 frequent mono- and bisyllabic words, and made semantic reading errors akin to deep dyslexia, for instance saying 'viande' (meat) for 'jambon' (ham). He could not read a single non-word or nonsense syllable, and out of the 26 letters of the alphabet he could only read A, B, D and Z.

In order to assess number reading, 137 digits or multidigit numbers were presented visually in Arabic script or as written verbal numerals, with either unlimited or 1-sec presentation time. Response time was unlimited and was recorded manually by the experimenter. For numbers under 20, N.A.U. counted on his fingers and/or verbally, from 1 up to about the correct value, a strategy that has been described in the context of 'right-hemisphere reading' in a left-hemispherectomized patient [36]. Reading time was thus linearly related to number magnitude (on the range 1–9: $r^2 = 99.6\%$, $P < 0.001$ for Arabic digits; $r^2 = 64.6\%$, $P < 0.01$ for verbal numerals; see Fig. 2). Errors, though less numerous with Arabic notation and with unlimited presentation time, always respected the approximate magnitude of the number read. Apparently, N.A.U. had an intact knowledge of the canonical counting sequence and used it to retrieve number names. Consistent with this analysis, N.A.U. could recite the overlearned number sequences 1, 2, 3, 4 . . . and 2, 4, 6, 8. . . , and used this knowledge to count sets of objects, but he could not count by 2 starting with digit 1 (1, 3, 5 . . .) nor count backwards (9, 8, 7 . . .). He also had considerable difficulties reading large 2-digit numbers. He attempted to read them by counting separately for the tens and units digits, but made many errors. Furthermore, the counting strategy was of no help when N.A.U. was confronted with the complex structure of some French 2-digit numbers (e.g. 73 = 'soixante-treize', literally 'sixty-thirteen'). Nevertheless, when he could not read a number, N.A.U. always proposed number names with plausible magnitude.

Since N.A.U. was perfect in reciting other automatic series such as months or days of the week, we expected that he might be able to read the corresponding words using the same strategy as with numbers [14]. Indeed N.A.U. made no error in reading twice the 7 days of the week presented in random order (median RT 3.7 sec). On each trial, he overtly recited the series starting with 'Monday' up to the word to be read. Reading time was linearly related to the rank in the series (Kendall's $\tau = 0.714$, $P = 0.0243$; see Fig. 2). N.A.U. was slower reading names of months (median RT 11.9 sec), and his response time was not correlated with the

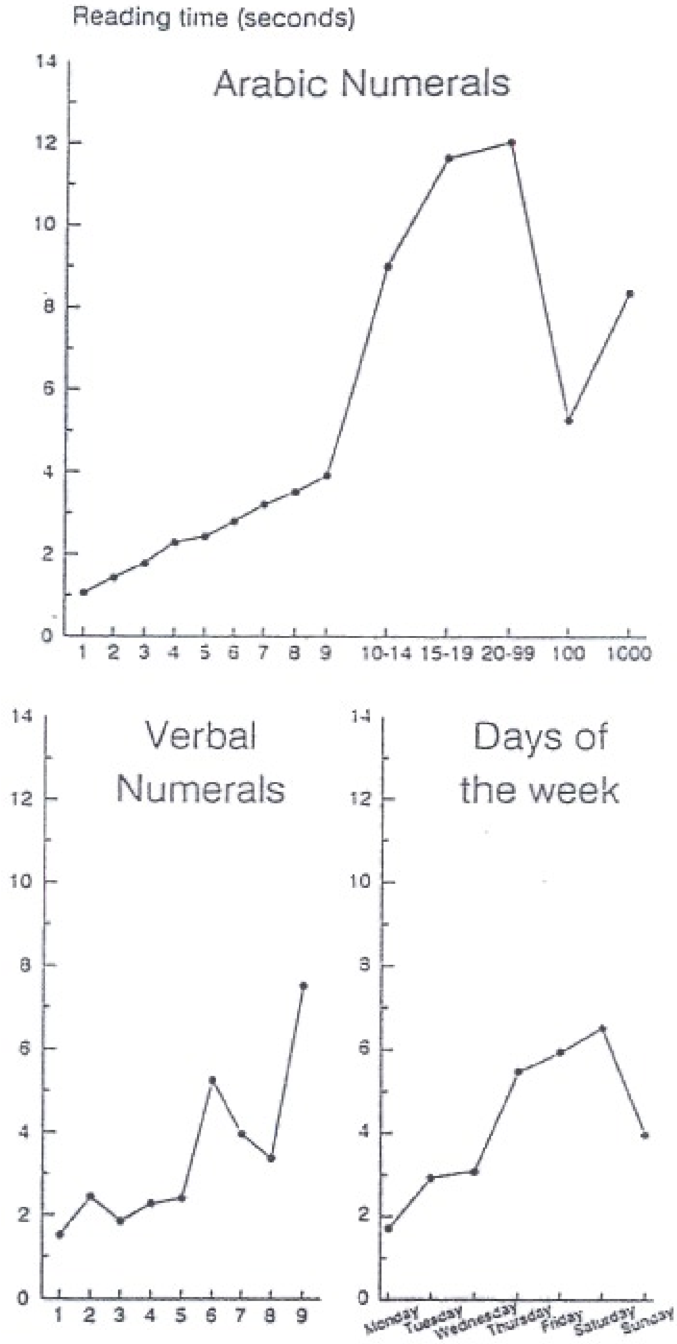

Fig. 2. Patient N.A.U.'s reading time for Arabic numerals, verbal numerals, and days of the week.

rank of the month. Still, his performance of 3 errors out of 12 trials was far better than with ordinary words.

CLINICAL ASSESSMENT OF NUMERICAL PROCESSING

N.A.U.'s preserved knowledge of the series of number words enabled him to count accurately when cards with 1–9 dots were presented for an unlimited duration. Even when the cards were displayed for about 1 sec, he was still able to provide a fair estimation of numerosity. However these abilities should not mask N.A.U.'s major acalculia. N.A.U. could not perform even the simplest arithmetical calculations, whether the problems were presented orally or visually. For instance he readily produced '3' in answer to '2 + 2'. In the course of his rehabilitation, he progressively developed a counting strategy for solving simple additions. He first counted up to the first operand, both verbally and on his fingers; then he resumed counting on his fingers from one up to the second operand, while continuing to recite the verbal series at the same pace, therefore reaching the correct sum. However this strategy was slow and painful. It was of no help for subtraction or multiplication problems. Even with additions, N.A.U. never used counting when he was urged to respond as fast as possible in the timed experiments 2–5 described below.

Number-related deficits were clearly not restricted to calculation. For instance when N.A.U. was asked several questions about common numerical facts, he made about 50% errors. He stated that January has 15 or 20 days, that a quarter of an hour is 10 or 20 min, that 1 hr is 50 min, that there are 5 seasons in the year, or that a year comprises 350 days. He even said that a dozen eggs was 6 or 10, despite the obvious similarity of the French words 'douzaine' (dozen) and 'douze' (12). In some cases he initially gave an erroneous response, such as that a hand has 4 fingers, or a horse 2 legs, and later corrected himself by counting.

N.A.U. was asked to name and sort a complete set of French coins and banknotes. Though his sorting was perfect, he had severe difficulties in naming. He even occasionally proposed numbers that are never used in coins or banknotes, such as 30 for a 20 F bill, or 15 for a 20 centimes coin. Nevertheless, N.A.U. never produced aberrant or bizarre numbers, as frontal patients sometimes do [38, 39]. As illustrated in the above examples, his responses were always approximately correct. N.A.U.'s preserved understanding of the adequacy of a number to a given situation was confirmed using a questionnaire originally designed by G. Deloche. N.A.U. was asked to judge whether a given number of items was adequate, too small, or too large, for a proposed real-world situation (e.g. 'Nine children in a school' = too small). He scored 9/9 correct in this test.

Thus, despite his severe alexia and acalculia, N.A.U.'s performance seemed correct as far as approximate numerical quantities were concerned. The suggested dissociation between exact and approximate number processing will now be evaluated in controlled experimental situations.

EXPERIMENT 1: ORAL AND WRITTEN ADDITIONS

Method

N.A.U. was presented with 15 additions using the digits 1–9, with totals ranging from 5 to 16, printed horizontally in Arabic digits. Fifteen other similar additions were also read aloud to him. N.A.U. was asked to produce verbally the result of each addition. When he hesitated or self-corrected, only his final response was considered.

Results and discussion

N.A.U. scored 14/15 correct with visually presented additions. He achieved this good performance by counting on his fingers in a low voice while fixating the digits. His only error could be accounted for by a failure of counting ($5 + 6 > 12$).

This counting strategy was apparently not available to N.A.U. for orally presented additions. He produced the correct result to only 4 of the 15 additions. However, his responses were always close to the correct result. Ten of his 11 errors were wrong by 1, 2 or 3 units. The remaining error was 6 + 7, to which he responded 8. He erred by larger amounts for larger additions, as attested by a significant correlation between the correct sum and the magnitude of his errors ($r = 0.497$, 13 d.f., one-tailed $P < 0.03$). Nevertheless, even small additions yielded some errors (e.g. $2 + 2 > 3$).

EXPERIMENT 2: MULTIPLE CHOICES IN ADDITION, SUBTRACTION AND MULTIPLICATION

Experiment 1 disclosed a dissociation between written additions, which were solved accurately by counting, and orally presented additions, for which N.A.U. apparently did not count and produced an approximate result. Experiment 2 was designed to evaluate the patient's preserved abilites and to compare his performance in addition, subtraction, and multiplication. An operation was presented and two results were proposed, one slightly incorrect and the other grossly incorrect. We assessed whether N.A.U. could find out the most plausible result.

Method

Twenty-seven problems were presented visually in horizontal form with two proposed results, one false by 1 or 2 units and the other grossly incorrect. The patient had to circle with a pen the most plausible result for each problem. The following problems were used:

Additions: 4+5=10/20; 1+2=4/9; 7+3=17/11; 3+5=4/9; 12+6=20/10; 15+35=28/48; 9+9=13/19; 20+9=31/41; 3+8=9/5

Subtractions: 4−1=2/9; 6−3=4/8; 9−2=2/6; 16−6=5/9; 13−4=10/20; 18−1=12/19; 40−9=28/20; 25−14=19/12; 9−8=2/7

Multiplications: 3×3=10/18; 5×4=19/11; 2×4=20/10; 3×6=20/40; 6×9=49/19; 7×5=63/43; 9×2=17/11; 4×8=19/29; 5×5=52/32

The patient performed this test twice with a 10-month interval.

Results and discussion

N.A.U. was fast and accurate with additions (17/18 correct, χ^2 (1 d.f.) = 14.2, $P < 0.0002$), although he apparently did not count. In fact, he often thought that the result he circled was actually the *correct* result of the addition! With subtractions and multiplications, N.A.U. hesitated more and sometimes said that he responded randomly. He indeed performed at chance level with subtractions (10/18 correct), but tended to do better with multiplications (13/18 correct, χ^2(1 d.f.) = 3.56, $P = 0.059$). In brief, N.A.U. could apparently select a plausible result for additions and perhaps also for multiplications, but not for subtractions.

EXPERIMENT 3: VERIFICATION OF WRITTEN ADDITIONS

The multiple-choice task used in experiment 2 permitted only a limited exploration of the patient's preserved abilities, since it did not easily allow for response time measurements, and it did not strictly preclude counting. For these reasons, we moved to a timed verification task in which the patient had to decide whether a given operation was true or false. Experiments 3, 4 and 5 explored N.A.U.'s addition abilities, and experiment 6 explored his multiplication abilites.

Method

Presentation of visual stimuli and measurements of response times were performed using a portable computer. Across several sessions, a total of 178 additions were presented visually in horizontal form (e.g. '2+3=9'). Each session started with four training problems, the results of which were not analysed. N.A.U. was asked to tilt a joystick to the right if the addition was correct and to the left if it was false. Table 1 presents the design of the additions problems. There were 24 additions with 1-digit operands and no carry (e.g. '3+4=7'). In this situation, the correct sum as well as the proposed sum were both 1-digit numbers. Seventy additions with 1-digit operands were also presented in which a carry operation was required (e.g. '5+6=11'). In this situation, both the proposed sum and the correct sum were 2-digit numbers. Finally, the remaining 84 additions comprised two 2-digit operands (e.g. '23+15=57'), did not involve carry computations, and the proposed sum was always a 2-digit number.

Table 1. Verification of additions (experiment 3)

Addition type	Correct		Proposed result False by 1–2		3–9		10–15		>40	
1-digit no carry	78%* 9‡	(3.1†)	86% 7	(2.9) (3§: 5‖)	13% 8	(2.0) (4: 5)	—		—	
1-digit with carry	60% 25	(3.0)	40% 10	(2.8) (5: 5)	24% 25	(4.2) (13; 13)	0% 10	(1.0) (10: 5)	—	
2-digit no carry	70% 27	(4.3)	46% 13	(4.3) (6: 6)	30% 13	(3.7) (6; 6)	54% 13	(3.6) (5; 6)	0% 18	(2.2) (10; 5)

*Per cent responses 'the addition is correct'.
†Median response time (sec).
‡Number of items.
§Number of items for which the proposed sum is larger than the correct sum.
‖Number of items for which the parity of the proposed sum differs from the parity of the correct sum.

Overall as well as within each testing session, about two-thirds of the additions (65.8% overall) were false. For false additions, the degree of falsehood was controlled by systematically varying the numerical distance between the proposed sum and the correct sum (Table 1). Two other parameters were controlled as much as possible within each cell of the design. First, half of the time, the proposed sum was larger than the correct sum. Second, the parity of the proposed sum matched the parity of the correct sum in half of the false problems. The exact distribution of problems with respect to these two parameters is given in Table 1.

Results

The overall median response time was 2.96 sec, and the error rate was high (30.3% errors). The patient denied using a counting strategy, and indeed we did not observe his usual vocal and manual counting behaviour. Median response time and percentage of additions that N.A.U. thought to be correct appear in Table 1. N.A.U. classified as correct 68.9% of the correct additions, a performance significantly higher than chance (χ^2 (1 d.f.)$=8.67$, $P=0.0032$). But this actually reflected response bias, since he also classified as correct 53.3% of the additions that were *incorrect* by 1 or 2 units. These two percentages did not differ (χ^2 (1 d.f.)$=2.10$, $P=0.148$; separate tests for each type of addition were also nonsignificant). Hence, N.A.U. could not discriminate correct and slightly incorrect problems, an observation which confirms his severe inability to compute additions.

However, N.A.U.'s performance was better when the proposed sum was more distant from the correct sum. With the 1-digit problems with no carry, he classified as correct 86% of the additions that were false by 1 or 2 units, but only 13% of those false by 3–9 units (χ^2 (1 d.f.)$=5.37$, $P=0.021$, Yates correction applied). He similarly showed an excellent performance (100% correct) for the two other types of additions when the proposed sum was the most remote from the correct sum (see Table 1). The effect of degree of falsehood was significant for 2-digit problems (χ^2 (3 d.f.)$=12.9$, $P=0.0049$), and marginally so for 1-digit problems with carry (χ^2 (2 d.f.)$=4.73$, $P=0.094$).

When the patient responded accurately, response times also tended to be faster (Table 1). This trend reached significance for 1-digit additions with carry (additions false by 10–15 units compared to the other three conditions: all one-tailed $P<0.011$) and for 2-digit additions (correct additions compared to additions false by more than 40: $t(43)=1.96$, one-tailed $P=0.028$). For 2-digit additions, the linear trend on RTs across the five levels of degree of falsehood was also significant [$F(1, 67)=4.76$, $P=0.032$].

Discussion

These results confirm those of experiments 1 and 2. N.A.U.'s performance deteriorated greatly when he had to attend to small numerical differences; he treated identically correct and slightly incorrect additions. However, he could rapidly and accurately reject grossly false additions. We conclude that N.A.U. possess an algorithm for fast but approximate evaluation of simple additions. The nature of this approximation process is explored in experiments 4 and 5.

EXPERIMENT 4: PROBLEM SIZE EFFECTS IN ADDITION VERIFICATION

When computing or verifying single-digit arithmetic operations, normal subjects are faster and more accurate with small numbers than with large numbers [1, 35, 44]. This *problem size effect* is very robust and is thought to reflect the time to access the relevant arithmetic facts in a memorized addition or multiplication table. Similarly, the time to perform a multidigit operation can be accurately predicted from the time to retrieve each of the elementary arithmetic facts required, plus some additional time for encoding and carry procedures [2, 20].

In experiment 4, we examine whether N.A.U.'s preserved ability to approximate additions can be accounted for by this model of normal processing. If N.A.U. computes additions by accessing a table of arithmetic facts, he should exhibit a problem size effect. For instance $1+2=3$ should be faster and easier to verify than $4+5=9$. Likewise for problems with 2-digit operands, $12+25=37$ should be faster and easier to verify than $45+32=87$. Furthermore, if N.A.U. computes multidigit additions digit by digit, additions with 2-digit operands should be about twice as slow as 1-digit additions. Such a trend was present in experiment 3 (see Table 1), and we wish to replicate it in a better controlled situation.

Method

One-hundred-and-nine addition problems were presented for verification using the same procedure and apparatus as above. Two subtests were designed, separating problems with 1-digit or with 2-digit results. In each subtest, problem difficulty was manipulated by systematically varying the size of the operands.

Subtest A. Operands, the correct sum, and the proposed sum, were all 1-digit numbers. Each operand could be either small (S) or large (L), resulting in four problem types: S+S (1+1, 1+2, 2+1, 2+2), L+S (5+1, 6+1, 5+2, 6+2), S+L (1+5, 1+6, 2+5, 2+6) and L+L (4+4, 4+5, 5+4). For each problem type, one-quarter of additions were correct. The remaining false additions were incorrect by either 1, 2 or 3 units. Four problems were presented for each combination of problem type and degree of falsehood, resulting in a total of 64 problems.

Subtest B. The correct sum and the proposed sum were always 2-digit numbers. Three levels of problem size were defined. For small problems (S), both operands were 1-digit numbers and a carry operation was needed. For medium problems (M), both operands were 2-digit numbers ranging from 10 to 29. Finally for large problems (L), the operands ranged from 30 to 59. M and L problems did not require carry operations. Within each set, one-third of problems were correct. Another third were problems false by 3 or 4 units, with the leftmost digit of the proposed sum always correct. Finally the last third were problems false by 13 or 14 units. Five problems were presented for each combination of problem type and degree of falsehood, resulting in a total of 45 problems.

Results

Median response time and percentage of additions that N.A.U. thought to be correct appear in Tables 2 and 3.

Subtest A. N.A.U.'s overall error rate was 34%, which is better than chance [χ^2 (1 d.f.) = 6.25, $P=0.012$]. Error rate did not differ across problem types [χ^2 (3 d.f.) = 0.28; see Table 3]. The tendency for larger problems to yield slower responses was far from significant. On the other hand, the effect of the degree of falsehood was again replicated. The percentage

Table 2. Effect of problem size in addition verification (experiment 4a)

Addition type*	Overall		Proposed result Correct		False by 1		2		3	
S+S	38†	(3.1‡)	100§	(2.2)	75	(2.6)	50	(4.0)	25	(3.7)
L+S	31	(3.8)	75	(4.0)	75	(5.7)	25	(4.1)	0	(3.4)
S+L	31	(5.0)	50	(4.8)	25	(5.3)	25	(3.7)	25	(3.6)
L+L	38	(4.6)	75	(3.0)	50	(4.8)	75	(5.2)	0	(6.5)
Overall	34	(3.9)	75	(3.0)	56	(5.3)	44	(3.8)	13	(3.8)

*S = small 1-digit operand (1 or 2). L = large 1-digit operand (4, 5 or 6).
†The column marked 'overall' gives the percentage of errors for each addition type.
‡Median response time (sec).
§Other columns give the percentage of responses 'the addition is correct'.

Table 3. Effect of problem size in addition verification (experiment 4b)

Addition type*	Overall		Proposed result Correct		False by 3–4		13–14	
S	20†	(3.2‡)	80§	(3.2)	40	(4.4)	0	(2.3)
M	67	(6.9)	40	(6.5)	80	(6.9)	60	(7.7)
L	33	(6.5)	80	(10.3)	60	(6.5)	20	(4.5)
Overall	40	(4.8)	67	(6.5)	60	(5.2)	27	(4.1)

*S = small (1-digit operands). M = medium (2-digit operands ranging from 10 to 29). L = large (2-digit operands ranging from 30 to 59).
†The column marked 'overall' gives the percentage of errors for each addition type.
‡Median response time (sec).
§Other columns give the percentage of responses 'the addition is correct'.

of problems that N.A.U. judged to be correct smoothly decreased from 75 to 13% as the degree of falsehood increased from 0 to 3 [χ^2 (3 d.f.) = 13.3, $P = 0.004$].

Subtest B. The overall error rate of 40% did not differ from chance [χ^2 (1 d.f.) = 0.91). However, error rate differed across problem types [χ^2 (1 d.f.) = 7.22, $P = 0.027$]. One-digit problems (S set) were easier than M and L problems, and were the only set to be classified significantly better then chance [χ^2 (1 d.f.) = 5.4, $P = 0.02$]. S problems were also classified faster then M and L problems [respectively $F(1, 24) = 11.7$ and $F(1, 24) = 12.0$, $P < 0.005$]. The effect of degree of falsehood was only marginally significant [χ^2 (2 d.f.) = 5.51, $P = 0.063$]. N.A.U.'s response choices differed only between correct problems and problems false by 13–14 units (67% vs 27%; χ^2 (1 d.f.) = 4.82, $P = 0.028$).

Discussion

The effect of degree of falsehood was again replicated in experiment 4. However no effect of problem size was found. First, for 1-digit additions without carry, neither error rate nor reponse time seemed to be affected by the size of the first and of the second operands. Second, if anything, 1-digit additions with carry, though tested in a different experimental session, tended to be even faster and more accurate than 1-digit additions with carry. Finally for additions of 2-digits operands, large problems were slightly but not significantly faster and less error prone than small problems, a tendency opposite to the normal problem size effect. In brief, the procedure enabling N.A.U. to reject grossly incorrect additions does not seem to be affected by problem size, and therefore differs from the addition procedure used by normal subjects.

The patient was however similar to normal subjects in at least one respect: as predicted, he was about twice as slow for additions of 2-digit operands than for additions of single digits (Table 3). One possible interpretation is that whatever algorithm N.A.U. uses for single-digit additions, he applies it twice when dealing with 2-digit operands. For instance when presented with '12+45=57', he would separately evaluate 1+4 and 2+5, thus taking twice as long to respond.

Normal subjects, however, do not necessarily have to compute these two additions. They generally adopt a self-terminating verification strategy by first computing the addition of the rightmost digits, then checking the result against the rightmost digit of the proposed sum, and moving to the leftmost digits only if this primary check fails [20]. There are indications in Tables 1 and 3 that N.A.U. did not employ this strategy. For instance in subtest B of experiment 4, N.A.U.'s performance dramatically improved from problems false by 3–4 (60% errors) to problems false by 13–14 (27% errors). The only difference between these two sets of problems lies in the leftmost digit of the proposed sum which is correct in one case and incorrect in the other. Similarly experiment 3 disclosed a large effect of the falsehood of the leftmost digit on N.A.U.'s judgements. This would suggest that N.A.U. bases his computation only on the leftmost digits of the operands and the proposed sum. For instance, when presented with the problem '31 + 12 = 59', he would compare 3 + 1 with the proposed 5, and would not take into account the units digits. This hypothesis is evaluated in experiment 5.

EXPERIMENT 5: ROLE OF THE LEFTMOST DIGIT IN ADDITION VERIFICATION

In experiment 5, we separately manipulated the falsehood of the left and right digits in additions of 2-digit numbers. If N.A.U. processed only the most significant digits in additions of 2-digit numbers, then his performance should be affected only by the falsehood of the leftmost digit. Alternatively, he might also be sensitive to the rightmost digit, in which case his performance would depend on the global distance between the correct sum and the proposed sum. We therefore defined and contrasted two different measures of the degree of falsehood: *D*1 (numerical distance between the correct sum and the proposed sum) vs *D*2 (numerical distance between the leftmost digit of the proposed sum and the sum of the leftmost digits of the operands).

Method

Across several sessions, 225 addition problems were presented for verification using the same procedure and apparatus as in the previous experiment. Three subtests were designed in order to separate the effects of *D*1 and *D*2.

Subtest A. *D*1 was systematically varied while *D*2 remained equal to zero. Only additions of 2-digit operands, with no carry, and whose sum was less than 40 were used. The leftmost digit of the proposed sum was always correct (*D*2 = 0). Ten additions were correct (*D*1 = 0; e.g. 10 + 11 = 21), 10 were false by 1–2 units (e.g. 25 + 14 = 38), and 10 were false by 6–9 units (e.g. 14 + 25 = 31). As appears in the above examples, false problems were paired across the two categories, with reversal of the order of the operands.

Subtest B. *D*1 was systematically varied while *D*2 remained equal to 1. Only additions of 2-digit operands, with no carry and whose sum was less that 70 were used. Fifteen additions were correct (*D*1 = 0; e.g. 25 + 34 = 59). Thirty other problems were false with *D*2 = 1. Fifteen of these were false by 1–4 units (1 ≤ *D*1 ≤ 4; e.g. 25 + 34 = 61), and 15 others were false by 16–19 units ($16 \leq D1 \leq 19$; e.g. $34+25=41$). As appears in the above examples, false problems were paired across the two categories, with reversal of the order of the operands, and conservation of the one-digit of the proposed sum. Finally parity and magnitude relations between the proposed and the correct sums were counterbalanced within each set of problems.

Subtest C. $D2$ was systematically varied for false additions, while $D1$ was kept approximately constant. Additions of 2-digit operands, with carry, and whose sum was less than 40 were used. Ten additions were correct ($D1=0$; e.g. $16+15=31$), and 20 were false. Ten of these had the leftmost digit of the proposed sum equal to the sum of the leftmost digits of the operands ($D2=0$; e.g. $18+17=24$); for the other 10, the leftmost digit of the proposed sum exceeded the sum of the leftmost digits of the operands by 2 units ($D2=2$; e.g. $17+18=46$). In both sets, the problems were paired, with reversal of the order of the operands, and $D1$ was approximately equal to 10 (range 9–12).

Subtests A and C were run three times each, whereas subtest B could only be run once.

Results

Median response time and percentage of additions that N.A.U. thought to be correct appear in Table 4.

Table 4. Role of leftmost digits in addition verification (experiment 5)

	Correct	Proposed result Slightly false	Very false
	Falsehood defined globally ($D1$* varied, $D2$† constant)		
Subtest A:	66.7%‡ (5.0§) $26+13=39$‖	50.0% (9.3) $25+14=38$	43.3% (7.8) $14+25=31$
Subtest B:	26.7% (4.7) $21+11=32$	13.3% (1.9) $11+21=29$	26.7% (4.3) $21+11=49$
	Falsehood defined on the leftmost digit ($D1$ constant, $D2$ varied)		
Subtest C:	63.3% (6.3) $25+17=42$	46.7% (8.4) $16+28=33$	6.7% (3.1) $28+16=55$

*$D1$ is the numerical distance between the correct sum and the proposed sum.
†$D2$ is the numerical distance between the leftmost digit of the proposed sum and the sum of the leftmost digits of the operands.
‡Per cent responses 'the addition is correct'.
§Median response time (sec).
‖Sample verification problem.

Subtest A. These were problems where the leftmost digit of the proposed sum was always correct. Hence, correct performance could only be attained by considering the ones-digits. N.A.U.'s performance did not differ significantly across the three conditions of the test [χ^2 (2 d.f.)$=3.48$, $P=0.18$]: correct, slightly incorrect and largely incorrect problems were classified with 42.2% errors, a percentage not better than chance [χ^2 (1 d.f.)$=2.18$, $P=0.14$]. However, responses were significantly slower with slightly incorrect problems than with correct ones [$F(1, 58)=10.6$, $P<0.002$]. This suggests that N.A.U. paid some attention to the ones-digits, but that it did not influence his final decision as to the correctness of a problem.

Subtest B. These were problems where the leftmost digit of the proposed sum was false by one unit, and the numerical distance between the correct sum and the proposed sum ($D1$) was either small or large (e.g. $25+34=61$ vs $25+34=41$). If N.A.U. attended only to the leftmost digits, he should perform identically with these two sets of false problems, even though $D1$

differed largely between them. Indeed, N.A.U.'s performance did not vary across conditions [χ^2 (2 d.f.) = 1.03, $P = 0.60$]. Over correct and incorrect problems he made 37.7% errors, a percentage again not different from chance [χ^2 (1 d.f.) = 2.69, $P = 0.101$]. He had a systematic bias towards responding that the problems were false (77.7% of his responses). Response time did not differ significantly between conditions. Thus, N.A.U. apparently verified additions with 2-digit operands on the basis of the leftmost digits.

Subtest C. This subtest was designed to provide positive evidence that N.A.U. relied predominantly on the leftmost digits. In false problems, the leftmost digit of the proposed sum was always false by one unit. But in half of the problems it was equal to the sum of the leftmost digits of the operands (e.g. $18 + 17 = 24$), whereas in the other half it was very different (e.g. $17 + 18 = 46$). If N.A.U. did not pay attention to the carry operation required by the ones-digits of the operands, he would perform very differently with these two sets of problems. Indeed, N.A.U. made 46.7% errors in the former condition and only 6.7% in the latter [χ^2 (1 d.f.) = 12.3, $P < 0.001$]. He noticed the falsehood of the additions, and hence performed better than chance ($P < 0.001$), only when the leftmost digit of the proposed sum differed from the sum of the leftmost digits of the operands. His response times were also significantly faster with this set of problems relative to the correct problems [F (1, 58) = 16.8, $P < 0.001$] and relative to the other set of false problems [F (1, 58) = 25.1, $P < 0.001$].

Discussion

N.A.U. noticed the falsehood of additions with 2-digit operands only when the leftmost digit of the proposed sum differed from the one he expected. The falsehood of the rightmost digit did not influence response choice, even though it apparently affected response time in subtest *A*. Experiment 5 therefore indicates that when the leftmost digit of the proposed sum is correct, N.A.U. may spend some time additionally considering if the rightmost digit is correct or not. But when the proposed leftmost digit looks sufficiently incorrect, N.A.U. rapidly classifies the addition as false.

GENERAL DISCUSSION OF ADDITION ABILITIES

Experiments 1–5 have documented a striking dissociation in patient N.A.U.'s addition abilities. On the one hand, N.A.U. always provides numbers of correct magnitude in response to an addition problem (experiment 1). He can choose the most plausible result for a given addition (experiment 2), and he easily detects when a 1- or 2-digit addition is grossly false (experiments 3–5). On the other hand, N.A.U. produces erroneous answers to even the simplest additions (e.g. $2 + 2 = 3$; experiment 1), and he does not differentiate between correct and slightly incorrect problems (e.g. $2 + 2 = 4$ vs $2 + 2 = 3$; experiments 3–5).

Is N.A.U.'s addition routine normal?

N.A.U. generally produces incorrect verbal answers to simple additions. However, the addition verification task, in which N.A.U. showed similar difficulties, does not require verbal production. Therefore N.A.U.'s deficit cannot be explained solely by an impairment of the number production system. Two other explanations might account for the observed dissociation between impaired exact calculation and preserved approximation abilities. First, the deficit may concern only a comprehension component. N.A.U. may fail to access the correct internal representation for the operands of an addition, but otherwise utilize intact classical addition routines on this noisy input. For instance the production of 3 in response to $2+2$ would result from the misperception of one of the operands for a 1. The second possibility is that N.A.U.'s addition algorithm itself is impaired, and that the addition routine he utilizes provides only noisy responses. For instance for $2+2$, this routine might output sometimes a 3, sometimes a 4, rendering accurate calculation or addition verification impossible.

The nature of N.A.U.'s addition routine was explored in experiment 4. As mentioned earlier, normal adult subjects exhibit a problem-size effect: the larger the operands of an arithmetic problem, the slower the time to compute the result. This robust effect is thought to reflect the retrieval of the result from a memorized table of arithmetical facts. If N.A.U.'s deficit is at a comprehension stage only, and if calculation itself is intact, N.A.U.'s response should obey the problem-size effect. However experiment 4 failed to disclose any effect of the size of the operands on N.A.U.'s response times and errors, suggesting that N.A.U. was not using a memorized addition table. In order to perform a more stringent test of this crucial question, all the 173 trials of 1-digit addition verification in experiments 3–5 were submitted to a stepwise regression analysis. Several variables that normally measure the problem-size effects in adults were introduced: a (the first operand), b (the second operand), $a+b$ (the correct sum), c (the proposed sum), $a*b$ (the product of the operands), and t (a dummy variable coding whether the problem was a "tie" ($a=b$) or not). Other variables included *split* (the absolute difference between c and $a+b$), *min* (the minimum of the two operands a and b), *carry* (a dummy variable coding if a carry operation was required or not), *sup* (a dummy variable coding $c>a+b$ vs $c<a+b$), *parity* (a dummy variable coding whether the parity of c was equal or not to the parity of $a+b$), *truth* (a dummy variable coding whether the addition was true or false), and $|a-b|$ (the absolute difference between a and b). Of all these variables, only variable *split* had a significant influence on response times ($P=0.0016$), confirming that N.A.U. was faster with grossly false problems. But there was absolutely no influence of the size of the operands or of their sum on response times.

A similar pattern of results was obtained in a stepwise regression on N.A.U.'s reponse choices ('this is a correct addition' vs 'this is an incorrect addition'). Only two variables had a predictive value. The effect of *split* ($P=0.0001$) confirmed that N.A.U. noticed the falsehood of an addition only when the proposed sum was distant from the sum. The effect of variable *sup* ($P=0.0053$) demonstrated an additional bias for responding that an addition was false when the proposed sum was larger than the correct sum (this can also be seen as a tendency to underestimate addition results; see below). Strikingly, the actual true–false status of a problem (variable *truth*) had no impact on N.A.U.'s judgements: he apparently judged the correctness of an addition solely on the basis of the distance between the correct sum and the proposed sum.

In brief, neither N.A.U.'s response times nor his errors showed the problem size effect so common in normal performance. Tie problems (e.g. $2+2$), which are among the easiest for normal subjects, were not faster or more accurate for patient N.A.U. These results suggest that N.A.U. was not accessing a memorized addition table. Similar arguments indicate that he did not use counting either. His fast responses to grossly false problems such as $2+2=9$ were at odds with his slow and easily recognizable counting behaviour. Indeed he denied counting in such situations. Furthermore in stepwise regression analyses, performance appeared not to be affected by operand size or by the minimum of the operands, as counting models would predict. Therefore our first hypothesis about N.A.U.'s deficit—impaired number comprehension with intact standard calculation routines—must be rejected. The results indicate that N.A.U. is not using a standard addition routine based on retrieval or counting, but instead utilizes an approximation routine discussed below.

Addition by activation of a candidate set

Figures 3 and 4, which summarize N.A.U.'s performance in addition verification pooled over all trials of experiments 3–5, provide some insights into this approximate calculation routine. Figure 3 gives the percentage of trials in which an addition was judged to be correct, as a function of the distance between the proposed sum and the correct sum. It is apparent that across different trials, N.A.U. readily accepted several different results as correct. For instance for single-digit additions without carry ('$a+b=c$', where a, b, c are 1-digit numbers), the correct result $c=a+b$ itself was judged as correct on 76% of trials, but $c=a+b-1$ was also judged as correct on 84.6% of trials, and $c=a+b-2$ on 60% of trials. This suggests that instead of accessing a single addition result, N.A.U. activated a whole set of plausible candidates. The asymmetry of the curves in Fig. 3 suggests that this acceptable set is slightly shifted in the direction of small numbers, and therefore that N.A.U. slightly but consistently underestimates addition results.

Finally Fig. 3 also reveals an enlargement of the acceptable set for larger additions. The enlargement for additions of 2-digit numbers can be attributed to the processing of leftmost digits only, as demonstrated in experiment 5. For instance N.A.U. classified $43+21=69$ as correct because he attended only to the decades-digits addition ($4+2=6$), and not to the units-digits addition ($3+1=9$) which would have been classified as false if presented in isolation. However the enlargement of the acceptable set from single-digit additions without carry to single-digit additions with carry cannot be explained in the same way. It seems that the variance of N.A.U.'s approximation increases for instance from $2+2$ to $6+7$. This effect should not be confounded with the problem size effect previously discussed. Total error rate does not increase, nor do responses become slower. The effect bears more similarity to Weber's law: internal variance increases with the magnitude of the addition processed. Several psychophysical studies indicate that the representation of numerical magnitudes in normals is 'compressive' and obeys Weber's law, i.e. larger numbers receive a coarser mental representation than smaller ones (for review see [25]). Weber's law is also characteristic of animal numerical cognition [18]. Therefore, although the observation of Weber's law in the context of addition approximation is novel, this aspect of patient N.A.U.'s results is coherent with several other reports.

For 1-digit additions, the response-time curves plotted in Fig. 4 confirm the suggestion that N.A.U. accesses a whole cloud of potential candidates for the result of the addition. N.A.U.'s fastest responses were to grossly incorrect additions like $2+2=9$, but his slowest responses were *not* for slightly incorrect additions like $2+2=3$, which he regularly classified as correct. Rather, the slowest responses, which appear as sharp peaks in Fig. 4, were for intermediate problems like $2+2=6$ or $2+2=7$. This pattern is easily understandable if N.A.U. activated a whole set of plausible results. When the proposed results fell well within the plausible set for a given addition, or conversely when it fell grossly outside this set, N.A.U. rapidly responded accordingly. The real difficulty for him was, as it should be, when the proposed result fell at the border between plausible and implausible results.

Analogical encoding of numerical magnitudes

Restle [37] has proposed a model of mental addition in which the operands are encoded as line segments on a mental 'number line' and are added by mental juxtaposition of the

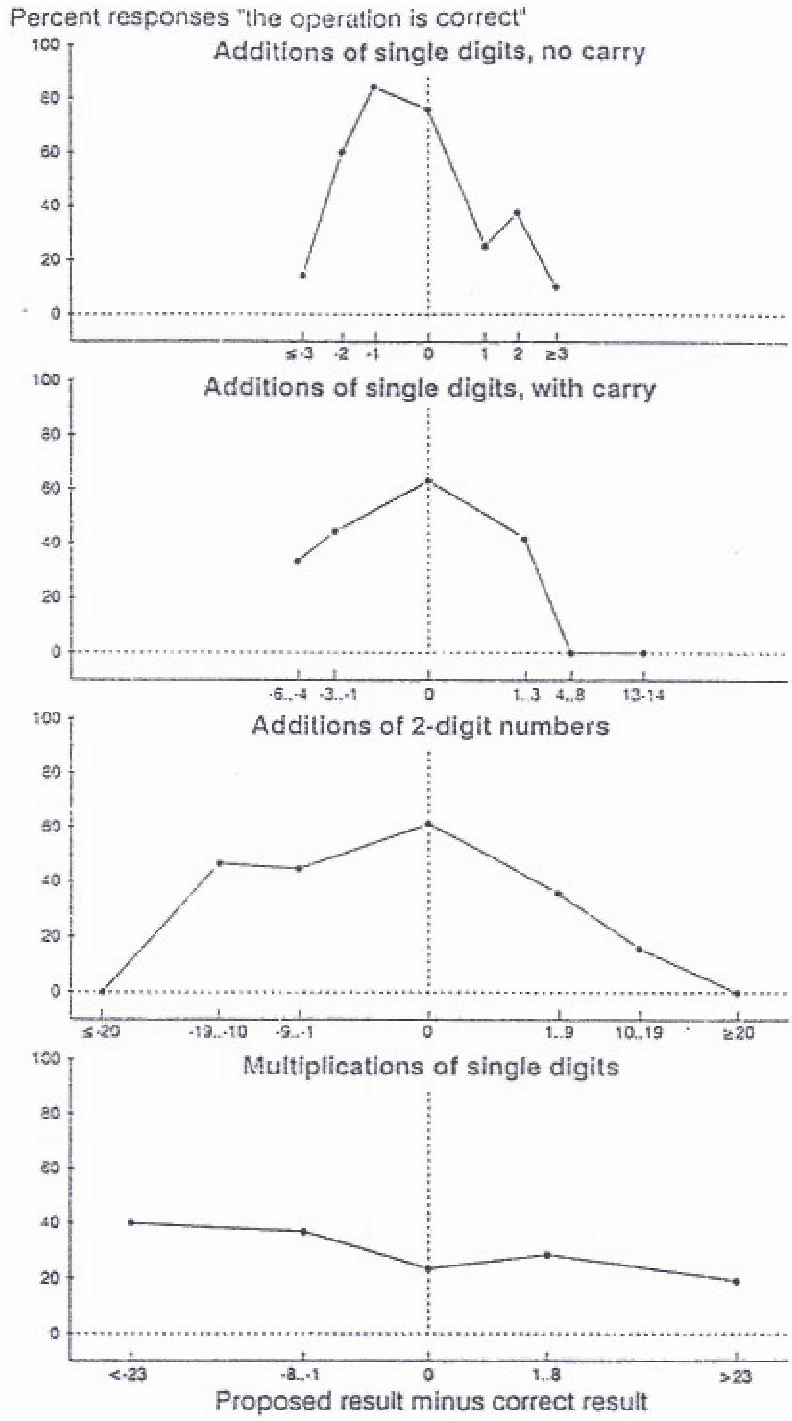

Fig. 3. Patient N.A.U.'s performance in verification of additions and multiplications. The X-axis scale is identical on all four plots and gives the log distance between the proposed result and the true result of the operation (0=correct; positive values indicate that the proposed result was too large). The peaks in the addition curves show that N.A.U. was unlikely to classify as correct a grossly false addition. A flat curve indicates random performance in multiplication verification.

segments. Although this model does not describe well the data from normal subjects performing exact addition [30], it can account for N.A.U.'s approximate evaluation of additions. The hypothesis that numbers are encoded analogicaly, in the same manner as purely physical magnitudes like line length, may explain the impossibility for N.A.U. to

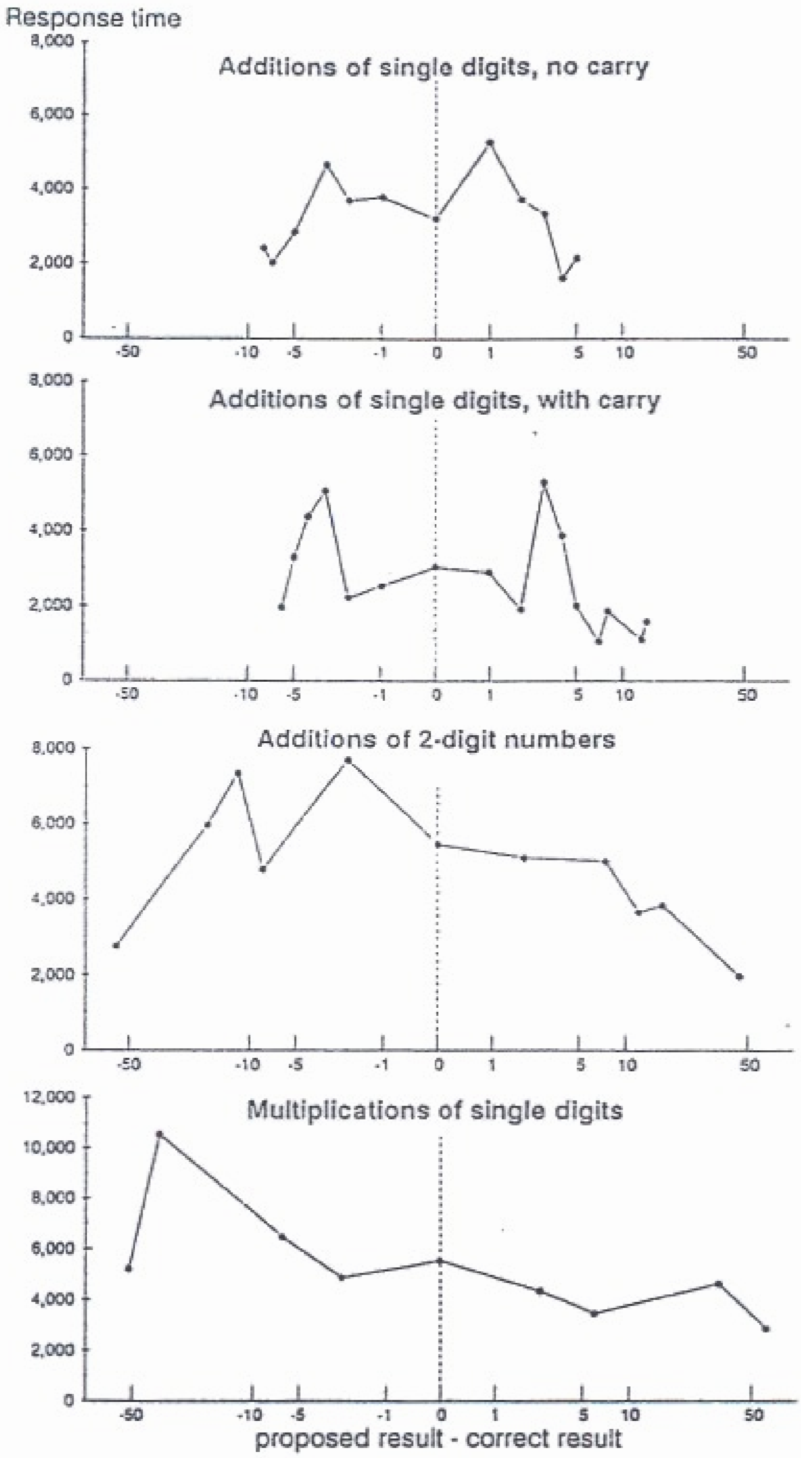

Fig. 4. Patient N.A.U.'s response times in verification of additions and multiplications. The slowest RTs roughly coincide with the distances for which N.A.U.'s response choices were the most variable (see Fig. 3). Same scale as in Fig. 3.

reach a perfectly accurate addition result. The intrinsic variability associated with the digital-to-analog transduction at the encoding would only allow for the representation and manipulation of approximate numerical magnitudes, not of exact numbers. Further, if number magnitude is mentally represented just like other physical dimensions, then it should obey Weber's law, and thus the internal variability is expected to increase for larger numbers.

This provides a simple interpretation of the decreasing precision with which N.A.U. can approximate larger and larger additions.

Approximation abilities and the analogical code in normals

Do N.A.U.'s approximation abilities represent a compensation strategy acquired after the loss of his exact calculation skills? Or do such approximation routines prevail covertly in most normal subjects, and were merely unveiled in N.A.U.'s case by the loss of all other numerical faculties? The *split effect* so prominent in N.A.U.'s addition data, also appears with normals: in addition verification, the more distant the proposed result is from the correct results, the faster the subjects classify the addition as incorrect [1, 2, 24, 47]. This distance effect is classically interpreted within a two-step calculate and compare model: subjects first determine the exact result, using standard symbolic algorithms, and then compare it with the proposed result. It is the comparison stage, not calculation, that is affected by the distance between the proposed sum and correct sum.

In some cases however, grossly incorrect additions are classifed so rapidly that subjects most likely do not have sufficient time to complete the exact calculation [2, 47]. This led Ashcraft and Stazyk [2] to suggest that 'a global evaluation process operates in parallel with (arithmetic fact) retrieval' (p. 185). Zbrodoff and Logan [47] also proposed that 'verification involves comparing the equation as a whole against memory' (p. 83), although this comparison process remained largely unspecified. We believe that patient N.A.U.'s data confer stronger plausibility to these proposals, and should prompt a more refined assessment of addition approximation in the normal subject.

The extent of patient N.A.U.'s deficit

N.A.U.'s addition performance suggests a complete loss of exact calculation routines, and a reliance on an analogical representation of numerical magnitudes. In the 'case report' section, evidence was presented that N.A.U.'s deficit is not limited to calculation only. In particular N.A.U. had an impaired memory for common numerical facts, and could only give approximate answers to queries about the number of eggs in a dozen, or the number of days in a month. In the following experiments, we assess the hypothesis that N.A.U. is completely unable to process exact numbers in digital or verbal form. We postulate that only his analogical number line is preserved, as well as the procedures which interface it with symbolic number notation systems. According to this hypothesis, N.A.U. should pass number-related tests only inasmuch as they require *approximate* number comprehension, production, memorization or manipulation; *exact* number processing should be impossible. Failure is therefore predicted for the following tasks tested below: verification of multiplications (experiment 6), short-term memory for the exact identity of Arabic digits (experiment 7), and parity judgement (experiment 10). Predictably feasible tests include short-term memory for approximate magnitudes (experiment 7), pointing onto a numerical scale (experiment 8), and comparison of Arabic numbers (experiment 9).

EXPERIMENT 6: VERIFICATION OF MULTIPLICATIONS

Following Restle [37], we supposed that N.A.U. was able to approximate additions by mentally juxtaposing two segments on his mental number line. Since no such strategy seems available for multiplication, we predict that N.A.U. should perform completely randomly in verifying simple multiplications.

Method

Across several sessions, a total of 120 multiplications of 1-digit numbers were presented visually in horizontal form (e.g. '8 × 4 = 28'). Four additional training problems were presented at the beginning of each session and their results were not analysed. N.A.U. was asked to tilt the joystick to the right if the multiplication was correct and to the left if it was false. Overall as well as within each testing session, about two-thirds of the additions (71.7% overall) were false. The degree of falsehood was controlled in two ways. First, for about half of the false problems, the proposed result did not belong to the multiplication table (e.g. '4 × 5 = 22'); for the other half, the proposed result belonged to the multiplication table (in 85% of cases it was a multiple of one of the two operands; e.g. '5 × 9 = 10'; '8 × 7 = 14'). Second, the proposed result could be either close to the correct result (absolute distance ≤ 8; e.g. '2 × 5 = 8'), or distant from it (absolute distance > 23, average absolute distance = 42.9; e.g. '7 × 2 = 56'). As before, two other parameters were controlled as much as possible within each cell of the design. First, half of the time, the proposed result was larger than the correct result. Second, the parity of the proposed result matched the parity of the correct sum in about half of the false problems. The exact distribution of problems with respect to these parameters is given in Table 5.

Table 5. Verification of multiplications (experiment 6)

	Correct	Proposed result False by 1–8	> 23
Proposed result within table	24%* (5.6†) 34‡	44% (4.5) 25 (13§; 13‖)	23% (4.7) 30 (16; 11)
Proposed result out of table		13% (4.2) 15 (8; 5)	38% (5.5) 16 (10; 7)

*Per cent responses 'the multiplication is correct'.
†Median response time (sec).
‡Number of items.
§Number of items for which the proposed result is larger than the correct result.
‖Number of items for which the parity of the proposed result differs from the parity of the correct result.

Results

The overall median response time was 4.20 sec and the error rate of 43.3% did not differ significantly from chance (χ^2 (1 d.f.) = 2.13, P = 0.144). N.A.U. has a strong bias towards responding that the multiplications were false (71.7% of his responses). However, the percentage of problems that he judged to be correct did not vary across the five categories of problems (χ^2 (4 d.f.) = 6.10, P = 0.192; see Table 5). Nor was performance affected by the distance between the correct and the proposed result [χ^2 (2 d.f.) = 0.73] by the correctness of the multiplication [χ^2 (1 d.f.) = 0.54], or by the fact that the proposed result belonged to the multiplication table or not [χ^2 (1 d.f.) = 0.45]. Similarly, response time did not differ across conditions.

Stepwise regressions were performed on response time and response choice to each individual problem. The independent variables used were *a* (the first operand), *b* (the second operand), *a**b* (the correct product), *c* (the proposed product), *split* (the absolute distance between c and a*b), *sup* (a dummy variable coding c > a*b vs c < a*b), *parity* (a dummy variable coding whether the parity of c was equal or not to the parity of a*b), *truth* (a dummy variable coding whether the multiplication was true or false), and *table* (a dummy variable coding whether the proposed result belonged to the multiplication table or not). For regressions on both response time and response choice, only variable *c* had a significant effect: when the proposed result increased, the patient responded faster and decided more often that the multiplication was false. The objective difficulty of the multiplication, as measured by variables *a*, *b* and *a***b*, did not affect N.A.U.'s performance.

Discussion

The patient was at chance level for classifying multiplications. He was not even able to reject grossly false results such as '$3 \times 3 = 96$'. He exhibited a similar bias in multiplication experiments than in addition experiments, i.e. he tended to respond that an operation was false when its proposed result was large. However this bias did not improve his results. We may conclude that N.A.U.'s approximation abilities are limited to addition and do not extend to multiplication.

EXPERIMENT 7: SHORT-TERM MEMORY

If the only representation that N.A.U. can access from the digital appearance of a number is its approximate magnitude, then he should not be able to memorize the exact identity of numbers over a certain time interval. We predict that he should only remember the approximate quantity that was presented to him, not the exact value. In a pilot experiment, we tested N.A.U.'s short term memory for a single Arabic digit. A digit in the interval 1–9 was presented for 600 msec, then the screen was blanked during 2.5 sec in the first session and during 10 sec in the second session. Finally a probe digit appeared and N.A.U. had to tell whether it was identical to or different from the one he just saw. N.A.U. was excellent in this task, scoring respectively 94.4 and 100% correct for the two sessions, and responding with median response times of 874 and 746 msec.

Of course in such a simple same–different task, it is possible to respond accurately using a low-level representation, for instance by comparing the visual appearance of the two digits. To avoid such low-level strategies and to better probe N.A.U.'s representational memory, we asked him to memorize sets of three consecutive digits. Since 3 was the value of N.A.U.'s digit span, we hoped that his short-term memory would be close to saturation. Indeed, N.A.U. made a number of errors which shed further light on his preserved abilities in number processing.

Method

On each trial, N.A.U. was visually presented first with a set of three consecutive digits in the range 1–9. The three digits appeared simultaneously, horizontally aligned for 1.2 sec. After a blank screen of 2.5 sec, a probe digit appeared and N.A.U. tilted the joystick to the right if the probe belonged to the previous set, and the left otherwise. Response time was recorded from the onset of the probe digit. For one-third of the trials, the probe digit actually belonged to the memorized set (e.g. 7, 6, 8 . . . 6). For another third, it was outside the memorized set by one unit (e.g. 7, 6, 8 . . . 9). For the remaining third, the probe digit was further away from the memorized set by at least four units (e.g. 7, 6, 8 . . . 2). For 'outside' trials, the probe digit was equally often smaller or larger than the memorized set.

N.A.U. was tested in three sessions, each including six initial training trials and six trials in each of the three experimental conditions. The same trials were used in the three sessions, but the left-to-right order of the digits composing the memorized set was varied: ascending for the first session (e.g. 6, 7, 8), random for the second session (e.g. 7, 6, 8), and descending for the third session (e.g. 8, 7, 6).

Results

The overall median response time was 1054 msec, and the error rate was 42.6%. Response latencies did not differ across conditions. N.A.U. also responded identically when the probe was within the memorized set and when it was just outside the set. In both cases, he chose the 'within' response in 61.1% (11/18) of trials, a performance not significantly different from chance. However the percentage of 'within' responses fell to 27.8% (5/18) when the probe was largely outside the memorized set. In other words, performance improved significantly [χ^2 (1 d.f.) = 4.05, $P=0.044$], and was marginally better than chance [χ^2 (1 d.f.) = 3.56, $P=0.059$), when the probe was more distant from the set.

Discussion

In the short-term memory task, as in previous addition verification experiments, N.A.U. exhibited again a striking dissociation between exact and approximate number knowledge. In addition verification, N.A.U.'s performance was governed by the degree of falsehood of the proposed sum. Likewise in the short-term memory task, his responses were determined by the proximity of the probe digit to the memorized set. He treated identically probes falling within vs slightly out of the set, but he rejected probes which were numerically more distant. It is not clear why N.A.U. could not retain in memory the exact identity of three Arabic digits, even for a duration as short as 2.5 sec. Either N.A.U. simply lacked the sufficient short-term memory, or he was unable to accurately encode the three digits of the set during the 1.2 sec display. Whatever the correct account, experiment 7 clearly demonstrated a contrasting residual ability to rapidly encode and retain in memory the approximate magnitude of a set of numbers.

Interestingly, MORIN, DEROSA and STULTZ [31] report that normal subjects also show an effect of number magnitude in the same memory task. Error rates were low, but the effect obtained on the response times. 'Outside' response times decreased with increasing numerical distance between the probe digit and the memorized set. Furthermore, when the set comprised consecutive numbers (e.g. 3, 4, 5, 6) and was stored on long-term memory, DEROSA and MORIN (1965, cited in Ref. [31]) found that 'within' responses were also affected by numerical distance: responses were slower to numbers in first and in fourth position (3 and 6 in our example) than to numbers in second or third position (4 and 5 in our example); numbers occupying the centre of the set were faster. This suggests that normal subjects, like patient N.A.U., are able to represent a memorized set as an activated region on the number line. Probes are harder to classify when they fall close to the boundary of this activated region. Naturally, normal subjects can also rely on symbolic memory to take their final decision, a strategy which was no more available to patient N.A.U.

EXPERIMENT 8: POINTING ON A NUMERICAL SCALE

We have postulated that N.A.U. understands what quantity a given number represents by activating an appropriate region on his mental number line. Therefore, given a number, whether written in Arabic notation or read aloud by the experimenter, N.A.U. should be able to point to its appropriate location on a vertical segment representing the interval 1–100.

Method

N.A.U. was presented with numbers distributed over the interval 1–100. Ten numbers were presented visually in Arabic notation, and ten others were read aloud by the experimenter. For each number, the patient was asked to point to the appropriate location on a vertical axis, 17 cm long, labelled '1' at the bottom and '100' at the top. The experimenter copied the chosen location onto a separate recording sheet, so that N.A.U. could not refer to his previous responses during the test.

Results and discussion

N.A.U.'s responses were fairly accurate (Fig. 5). For both oral and written presentation, the location to which he pointed was highly correlated with the correct location (respectively $r=0.965$ and 0.971, and in both cases $P<0.0001$). Overall, he made an average absolute error of 7.9 units (13 mm). In this task, normal subjects would probably be very accurate with numbers like 25, 50 or 75, which correspond to simple fractions of the scale range 100. However N.A.U. was not better with such numbers. For instance he placed 10 at 1/4 of the 1–100 segment, and 75 at 9/10 of it. It is likely that divisibility relations (e.g. that 75 is 3/4 of 100) were not available to him because of his severe calculation impairment. That his performance was nevertheless satisfactory is all the more striking and suggests a preserved representation of the relations between numerical magnitudes.

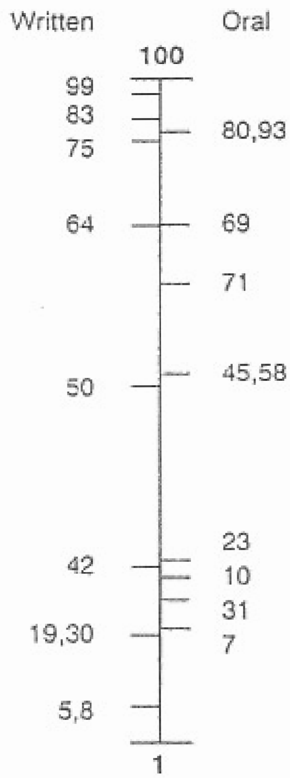

Fig. 5. The locations which patient N.A.U. indicated on a 1–100 scale in response to numbers presented visually or auditorily.

EXPERIMENT 9: NUMERICAL COMPARISON OF 1- AND 2-DIGIT NUMBERS

N.A.U.'s representation of numerical magnitudes was further probed by asking him to classify numbers as larger or smaller than a fixed standard of reference. In normal subjects, numerical comparison is thought to involve a transduction from the digital notation to an analogical magnitude representation, and the determination of the relative positions of the two numbers on the number line [11, 12, 23, 32, 33]. Since N.A.U.'s digital-to-analogical transduction abilities are presumably preserved, he should perform normally in this task. The inherent variability of the analogical representation might however yield some errors when the two compared numbers are close in magnitude (e.g. 4 vs 5).

Method

Across four sessions, 1- and 2-digit Arabic numbers were presented visually for comparison respectively with standard numbers 5 and 55. For each target number, N.A.U. tilted the joystick to the right if the number was larger than the standard, and to the left otherwise. For 1-digit numbers, all the numbers 1–9 except 5 were presented seven times each in random order, preceded by four training trials. For 2-digit numbers, all numbers 31–79 except 55 were presented once in random order, preceded by 10 training trials. Both sets of targets were used twice, once with a tachistoscopic presentation (300 msec duration), and once with a response-terminated visual display.

Results

N.A.U. responded fast (median RT 1005 msec) and made no error in the 56 trials of 1-digit number comparison with unlimited presentation duration. He made only one error to target 53 in the 48 trials of 2-digit number comparison with unlimited presentation (median RT 975 msec). With tachistoscopic presentation, error rate increased but performance remained excellent. With 1-digit numbers he made four errors (7.1%) to targets 3, 4, 4 and 6, and his median RT was 8000 msec. With 2-digit numbers he made four errors (8.3%) to targets 50, 52, 53 and 54, and his median RT was 814 msec.

Discussion

The patient's performance was virtually normal, a remarkable result given that his short response time would not have allowed him to read aloud the presented numbers or to use counting. N.A.U. could compare Arabic numbers and therefore could mentally represent their magnitudes, even with display durations of 300 msec. This experiment confirms that fast approximate encoding of numerical magnitudes is preserved.

The clustering of N.A.U.'s errors to target numbers close to the standards of comparison suggests that encoding precision was not perfect. However, the sparseness of comparison errors contrasts sharply with the high error rate observed in addition verification and in the short-term memory experiment. If N.A.U. is only able to compute the approximate value of numbers, how can he perform so well in numerical comparison? First, in numerical comparison, only a single target number is processed on each trial, whereas three numbers are processed in the addition task. Thus the error rate should be at least three times larger in the addition task, especially if the addition process itself generates errors.

Second, at the computational level, numerical comparison is demonstrably easier than addition verification or the short-term memory task. The latter two tasks belong to the general class of same–different tasks: *in fine*, they require the subject to decide whether two numbers are the same or not. In the appendix, we demonstrate mathematically that the same–different task cannot be performed on an analog magnitude representation without making systematic errors. Even if an optimal decision criterion is used, the subject is bound to respond 'same' to numbers that actually differ by only a small amount. By contrast, the optimal decision criterion for larger–smaller comparison is more efficient. We show that under the same conditions of variability, an ideal observer may respond wrongly that 4 is identical to 5 (e.g. that $2+2=5$) in 66% of trials in a same–different task, but nevertheless respond correctly that 4 is smaller than 5 in 80% of trials in a smaller–larger comparison task. These percentages are close to N.A.U.'s actual response rates. Our analysis therefore demonstrates that good performance in the numerical comparison task is compatible with the hypothesis of a high variability in the internal representation of magnitudes.

EXPERIMENT 10: PARITY JUDGEMENT

N.A.U.'s number knowledge was further explored by asking him to classify 1-digit numbers into odd and even numbers. This task of parity judgement is formally similar to numerical comparison: in both cases, numbers must be classified in one of two categories (smaller/larger, odd/even). On preliminary testing, N.A.U. gave a reasonable definition of the notion of parity, and he was able to laboriously classify 10 cards bearing the digits 0–9 into two piles of odd and even digits. Therefore one might expect N.A.U. to master the parity judgement task. However if our hypothesis is correct and if N.A.U. only encodes the approximate magnitude of numbers, then number n should be hard to distinguish from numbers n + 1 and n − 1. We would thus predict parity judgements to be defective.

Method

N.A.U. was presented with two lists, each comprising 11 initial training trials and 9 occurrences of each of the digits 0–9. Each target was presented visually and remained on the computer screen until N.A.U. made his parity judgement. In the first session, N.A.U. responded by tilting the joystick to the left if the target number was even, and to the right if it was odd. In the second session, response sides were reversed.

Results and discussion

Median response time was 2701 msec, and the overall error rate of 44.4% did not differ from chance [χ^2 (1 d.f.) = 0.89]. N.A.U. was distressingly aware of responding quasi randomly. The two sessions therefore had to be interrupted before completion, and only a total of 72 trials could be analysed. N.A.U. indeed responded randomly to odd numbers (55.3% errors). However he was slightly but significantly better than chance with even numbers [32.4% errors; χ^2 (1 d.f.) = 4.24, P = 0.040]. Note that N.A.U. was able to recite the verbal sequence 0, 2, 4, 6, 8 . . . , but not the sequence 1, 3, 5, 7, 9 . . . . This may explain his better performance with even than with odd numbers in parity judgement. Furthermore, the preservation of the rote sequence of even numbers presumably allowed him to correctly classify even vs odd digits in the preliminary card-sorting test.

In several numerical tasks, including parity judgement, normal subjects are faster and more accurate when responding with the right-hand key to a large number and with the left-hand key to a small number, than in the opposite condition [12, 13]. This effect is thought to reflect an automatic activation of the left-to-right oriented 'number line', or mental map of number magnitudes. Interestingly, patient N.A.U. exhibits this effect. Target numbers were classified as small (0–4) or large (5–9). Parity judgement trials requiring a leftward response to a small target or a rightward response to a large target were labelled as 'congruent'. The other trials were labelled as 'incongruent'. N.A.U. performed significantly better for congruent than for incongruent trials (31.3% vs 55.0% errors; χ^2 (1 d.f.) = 4.06, P = 0.044). In others words he was biased to press the right-hand key for a large number, and the left-hand key for a small number. This result indicates that as far as numerical magnitude is concerned, N.A.U.'s performance shows even the most subtle characteristics of normal processing. This contrasts sharply with the almost complete loss of parity knowledge.

GENERAL DISCUSSION

We have presented the case study of a severely aphasic and acalculic patient with selective preservation of approximation abilities. The dissociation between impaired exact processing vs preserved approximation was observed in several domains of numerical competence:

Number reading: N.A.U. showed considerable difficulties and had to count on his fingers, but his responses were always numbers of plausible magnitude.

Number comprehension: N.A.U. understood the relative magnitudes of numbers, as attested by his good performance in numerical comparison and in pointing onto a numerical scale. However, he was largely impaired in judging the parity of 1-digit numbers.

Memory: N.A.U. failed to memorize the exact identities of three digits for more than a few seconds. However he still remembered their approximate magnitude.

Calculation: N.A.U. was unable to solve simple additions, multiplications or subtractions. However he could accurately reject grossly incorrect additions such as $2+2=9$, or select the most plausible result for an addition.

Number knowledge: N.A.U. remembered only approximate facts, for instance that a year is *about* 350 days, or a month *about* 15 or 20 days.

Two mental calculation systems

These dissociations suggest the existence of two parallel routes in number processing, one for exact symbolic processing and the other for the manipulation of approximate numerical magnitudes. In adults, most numerical calculations are normally achieved by manipulations of symbols in digital notation [2, 20]. The disruption of the digital route in N.A.U. yields severe deficits. However, this route is supplemented by a second pathway specialized in the representation of approximate magnitudes in analog form. The analog representation or number line [37] is used in number comparison, memory for magnitudes, and generally all tasks requiring estimations of quantities, which are intact in patient N.A.U. Figure 6 summarizes the functions supposedly subsumed by each processing pathway.

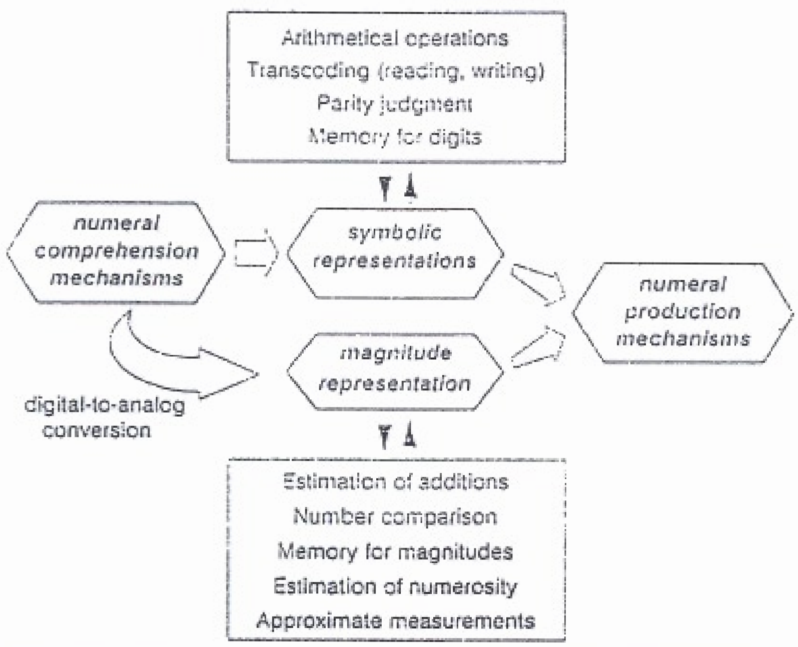

Fig. 6. Schematic diagram of the proposed dual-route model of number processing.

Is the assumption of a dual pathway really necessary, or might N.A.U.'s performance be explained within a single-route model? In McCloskey's model of number processing [7, 27, 28], for instance, one might assume that N.A.U.'s abstract internal representation has become inherently more variable following the lesion. Therefore his representation of, say, 45, which is normally {4}10EXP1 {5}10EXP0, would randomly switch to {3}10EXP1 {4}10EXP0 or to {4}10EXP1 {6}10EXP0, etc. Since the abstract internal representation allegedly intervenes in number comprehension, calculation and production, approximation errors would be predicted in all tasks. However, such a lesion would not by itself suffice to explain N.A.U.'s deficit. N.A.U. cannot retrieve any addition, multiplication or subtraction facts; nor can he apply standard multidigit calculation procedures. The possibility cannot be dismissed that these additional deficits stem from lesions at other sites within the functional architecture. However, our account in terms of a single destruction of symbolic number processing seems more economical. Furthermore the existence of an independent non-verbal number processing system is corroborated by animal and human infant data, as mentioned in the introduction.

Similar difficulties confront an explanation of N.A.U.'s deficit in CAMPBELL and CLARK's encoding-complex model [6]. In this model the various internal codes for numbers are interconnected by a network, with some links coding for parity information, others for verbal associations, and presumably others for proximity of numerical magnitude. To explain N.A.U.'s deficit, one should suppose that only the links specifying proximity of numerical magnitude were spared by the lesion. The encoding-complex model is sufficiently underspecified to allow for such a lesion, but the account would then become formally equivalent to our more constrained two-route explanation.

Converging evidence

In the adult, the two number processing routes are expected to work simultaneously and in coordinated fashion, and their respective roles may therefore be difficult to delineate. Nevertheless, some arithmetical tasks seem to tax selectively one or the other calculation system. In number comparison, response times and error rates are affected by the numerical distance between the two compared numbers (distance effect [5, 11, 12, 23, 32, 33]). In verification of arithmetic calculations, performance depends on the distance between the proposed result and the actual result of the operation (split effect [1, 2, 24, 47]). Finally in a short-term memory task, the time to judge if a probe digit belongs to a previously memorized set of digits depends on the distance between the probe and the set [31]. In our view, such distance effects indicate processing through the analog route. Conversely, there is evidence that normal subjects perform certain algorithmic calculations without any expectation of the approximate result. For instance in the initial stages of subtraction acquisition, errors such as 75–25=410 are produced [45], suggesting that subjects are 'blind' to the quantities associated with the computation. Indeed, it has been argued that number acquisition in children mostly consists in appropriately bridging the symbolic number processing algorithms and the preverbal magnitude representation [19].

The dual-route framework can also account for the performance of patient D.R.C. [46], which bears considerable similarity to patient N.A.U. Patient D.R.C. had severe difficulties in performing simple calculations, which could be traced back to a deficit in accessing memories for elementary arithmetical facts. However D.R.C. could rapidly provide an estimate of the operation result. For instance for 5+7 he replied '13 roughly'. Like N.A.U., D.R.C. had no difficulties in number comparison and in estimation of quantities. However unlike N.A.U., D.R.C. was also perfect in number reading, digit span, and number knowledge. D.R.C. probably had a mild and isolated deficit in arithmetic fact retrieval, and therefore sometimes had to rely on his preserved approximate calculation abilities. However he also had several other strategies at his disposal (e.g. parity checking). By contrast, our patient N.A.U. had a much more severe deficit affecting almost all symbolic number processing (reading, calculating, short- and long-term memory). Therefore the selective preservation of approximate number processing was exposed more distinctly to investigation.

Calculating with the right hemisphere?

Patient N.A.U. suffered from a massive lesion involving almost all of the posterior half of the left hemisphere (Fig. 1). This raises the issue of the contribution of the right hemisphere to his preserved abilities. N.A.U.'s reading was similar to that of the left-hemispherectomized deep dyslexic patient described by Patterson *et al.* [36]. Both N.A.U. and Patterson *et al.*'s patient produced semantic errors in word naming, could not read a single non-word, and could access the phonological form of printed words, for instance numbers, by reciting an overlearned verbal series. The suggested right-hemisphere mediation in N.A.U.'s reading might extend to calculation processes. The notion of 'calculating with the right hemisphere' is largely undermined by the preponderance of left-hemisphere lesions in acalculia (e.g. Ref. [22]). Nevertheless, studies with split-brain patients [41], patients with massive aphasia (e.g. Ref. [4]), as well as hemifield presentation data from normal subjects (e.g. Ref. [16]), have long suggested the existence of limited number processing abilities in the right hemisphere, which Assal and Jacot-Descombes [3] have called 'arithmetic intuition'. Whether right-lesioned patients would show a dissociation of calculation abilities opposite to patient N.A.U.'s remains to be evaluated.

Acknowledgments—This paper is dedicated to the memory of Prof. Jean-Louis Signoret, who first drew our attention to the potential interest presented by patient N.A.U.'s case. The research reported here would never have been accomplished without his sustained interest and encouragement. We also thank patient N.A.U. for his collaboration, and Jacques Mehler and the staff at L.S.C.P. for helpful discussions.

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APPENDIX: OPTIMAL STRATEGIES FOR SAME–DIFFERENT AND LARGER–SMALLER COMPARISON TASKS

Supposing that N.A.U. encodes numbers as fuzzy activated regions on a mental number line, how can he decide that two numbers are equal (same–different comparison), or choose which of two numbers is the larger (larger–smaller comparison)? In this appendix, we derive the statistically optimal choice strategies for the two tasks. We show that the larger–smaller comparison task can be performed with high accuracy even if the variance of internal representations is important. On the contrary, the optimal strategy for the same–different comparison task necessarily yields a systematic error (responding 'same' to slightly different numbers). This difference in optimal strategies for the two tasks may explain why on the one hand, N.A.U. was wrong in verifying statements such as '2+2=5' or '4 belongs to the set 5, 6, 7', while on the other hand he could reliably classify 4 as smaller than 5.

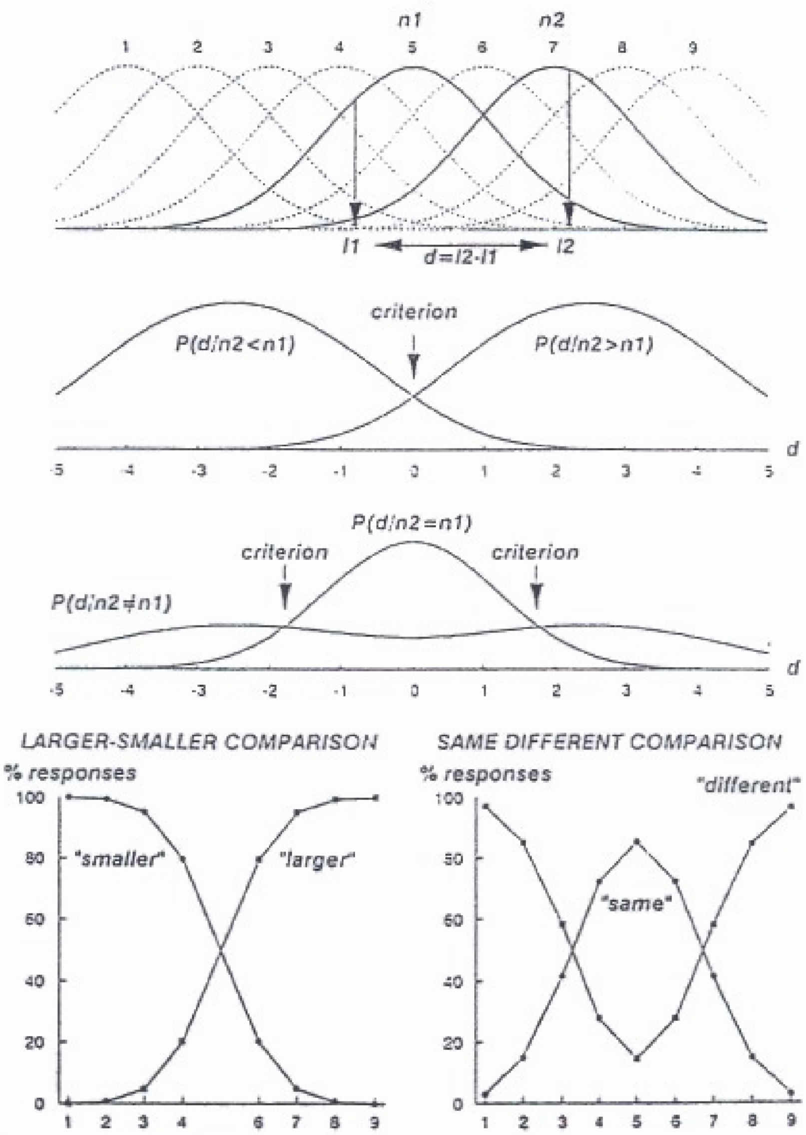

Fig. 7. Mathematical model of the internal encoding of numerical magnitudes and of the optimal strategies for larger–smaller and same–different comparison. See the appendix for details.

Suppose that numbers $n1$ and $n2$ are presented visually, and that N.A.U. must decide whether $n2$ is larger or smaller than $n1$ (larger–smaller comparison task), or whether $n1$ and $n2$ are same or different (same–different comparison task). In both cases, we assume that $n1$ and $n2$ are internally encoded at locations $l1$ and $l2$ on the number line. On each trial, the random variables $l1$ and $l2$ are drawn from two Gaussian distributions with fixed variance σ^2, centred on the appropriate locations for $n1$ and $n2$ on a linear scale (the hypothesis of linearity and of fixed variance on the number line are not crucial to the argument and are adopted here only for simplicity). In the numerical simulations below, we take $\sigma = 1.2$, which means that the 95% confidence interval for the representation of 5 is [2.6, 7.4]. Because of this high encoding variability, the internal representations are not always veridical. First, $l1$ may be smaller than $l2$ when $n1$ is actually larger than $n2$. Second, $l1$ will generally differ from $l2$ even if $n1$ is equal to $n2$. Erroneous responses therefore cannot be avoided, and the problem is to find strategies which minimize the error rates.

Smaller–larger comparison task

When a given internal difference $d = l2 - l1$ is observed, positive or negative, what is the optimal way of choosing between the two responses '$n2 > n1$' vs '$n2 < n1$'? The maximum likelihood principle states than one should respond '$n2 > n1$' if and only if the probability of observing d when $n2$ is larger than $n1$, $P(d|n2 > n1)$, is larger than the probability of observing d when $n2$ is smaller than $n1$ ($P(d|n2 < n1)$). Since the difference $l2 - l1$ is normally distributed with mean $n1 - n2$ and variance $2\sigma^2$, $P(d|n2 > n1)$ and $P(d|n2 < n1)$ are easily computed. This computation was performed in Fig. 7 in the case of a comparison of the target numbers 1–4 and 6–9 with standard 5. The two curves cross at $d = 0$, yielding the following fairly intuitive optimal criterion: one should respond '$n2 > n1$' if and only if a positive difference ($l2 > l1$) is observed. The expected error rate of this strategy can then be computed for each value of $n2 - n1$ (Fig. 7). This error rate can never exceed 50% (by hypothesis $P(l2 > l1|n2 > n1) > \frac{1}{2}$), and therefore systematic errors cannot occur. Assuming $\sigma = 1.2$, the error rate remains below 20.2%, this maximum value being reached only when the difference $n2 - n1$ is minimal ($|n2 - n1| = 1$). Thus, even with a noisy internal representation of quantities, it is possible to make larger–smaller comparisons with high accuracy.

Same–different comparison task

A similar analysis can be made for the same–different comparison task. The probability of observing a given difference d when $n1 = n2$, $P(d|n1 = n2)$, is given by a Gaussian with mean 0 and variance $2\sigma^2$. Conversely, $P(d|n1 \neq n2)$ is obtained by averaging the Gaussians for all values of $n2$ which differ from $n1$. The two density functions $P(d|n1 = n2)$ and $P(d|n1 \neq n2)$, plotted on Fig. 7, cross at two symmetrical points $-c$ and $+c$ which define the interval for the 'same' response. The optimal strategy is to respond '$n1 = n2$' whenever d falls within the interval $[-c, +c]$, i.e. whenever $l2$ is sufficiently close to $l1$.

For each target number $n2$, one may then calculate the probability of responding that it is the same as $n1$. A systematic error is found for $n2 = n1 \pm 1$: a 'same' response is ellicited on 72.3% of trials. This systematic error is required in the optimal strategy in order to ensure a reasonable hit rate (here 85.5% success), when $n2$ is actually equal to $n1$. Note that the exact value of the criterion c varies with the expected range of numbers $n2$, and with the *a priori* probabilities of 'same' and 'different' responses, here taken to be equal. Nevertheless, some criterion must be adopted and will necessarily yield some systematic errors at small numerical distances between the compared numbers.