Cognition, 44 (1992) 1–42

Varieties of numerical abilities

Stanislas Dehaene*
INSERM et CNRS, Laboratoire de Sciences Cognitives et Psycholinguistique, 54 Bd. Raspail, 75270 Paris Cedex 06, France

Abstract

Dehaene, S., 1992. Varieties of numerical abilities. Cognition, 44: 1–42.

This paper provides a tutorial introduction to numerical cognition, with a review of essential findings and current points of debate. A tacit hypothesis in cognitive arithmetic is that numerical abilities derive from human linguistic competence. One aim of this special issue is to confront this hypothesis with current knowledge of number representations in animals, infants, normal and gifted adults, and brain-lesioned patients. First, the historical evolution of number notations is presented, together with the mental processes for calculating and transcoding from one notation to another. While these domains are well described by formal symbol-processing models, this paper argues that such is not the case for two other domains of numerical competence: quantification and approximation. The evidence for counting, subitizing and numerosity estimation in infants, children, adults and animals is critically examined. Data are also presented which suggest a specialization for processing approximate numerical quantities in animals and humans. A synthesis of these findings is proposed in the form of a triple-code model, which assumes that numbers are mentally manipulated in an arabic, verbal or analogical magnitude code depending on the requested mental operation. Only the analogical magnitude representation seems available to animals and preverbal infants.

"What is a number, that a man may know it, and a man, that he may know a number?" (McCulloch, 1965). From Plato to Mill, Locke or Frege, the problem of the nature of numbers has always concerned philosophers of the mind. Although the issue has often been addressed on a logical-mathematical basis, the potential contributions of psychological experimentation to the comprehension of

Correspondence to: Stanislas Dehaene, INSERM et CNRS, Laboratoire de Sciences Cognitives et Psycholinguistique, 54 Bd. Raspail, 75270 Paris Cedex 06, France.

*I thank S. Frank, J. Mehler and M. McCloskey for helpful comments. This work was supported in part by INSERM.

the multiple facets of the number concept have recently been recognized (e.g., Kitcher, 1984). In parallel and somewhat independently, experimenters have begun to explore the mental processes used in number comprehension, production and calculation. Research in this area tends to cross the traditional boundaries of cognitive science. Studies of arithmetical competence in the adult benefit from an understanding of the developmental sequence through which numerical representations pass during childhood (Ashcraft, this issue; Gallistel & Gelman, this issue). Likewise, precursors of human numeracy are found in the remarkable sensitivity of some animals to numerical parameters (Gallistel & Gelman, this issue). Finally the study of brain-lesioned patients (McCloskey, this issue) and of gifted adults (Seron et al., this issue) sheds light on the modular architecture and range of individual variability of the human number processing system.

The purpose of this issue is to confront and compare the large quantities of data that have emerged from these domains in recent years. However, such interdisciplinary endeavours are often hindered by lack of a common reference vocabulary. The present introductory paper aims at providing the newcomer to the field with a selected review of the main experimental findings and current research issues surrounding the "concept of number".

This review will specifically focus on the relation between numerical abilities and the language faculty. The prevailing notion that human numerical activities are deeply linked to language is critically examined. First, number notations are described in a historical perspective. Then the mental processes used to comprehend numerals, perform mental calculations and produce an appropriate written or spoken numerical answer are reviewed. Formal symbol-processing models of calculation and transcoding have been extremely successful. However, an inclusive review suggests that there are other domains of numerical competence that cannot be so easily reduced to a subset of language abilities. The case for non-verbal quantification of sets of objects, including infant and animal counting as well as adult subitizing, is discussed in some detail. Experiments are also reviewed suggesting a non-verbal, analogical representation of numerical quantities in human adults. Dwelling on my earlier notion of "two mental calculation systems" (Dehaene & Cohen, 1991), I will conclude with some speculations on a model for the interaction between preverbal and verbal numerical abilities in human adults.

I. THE CLASSICAL ADULT MODEL: TRANSCODING AND CALCULATION

For the lay person, calculation is the numerical activity par excellence. Calculation in turn rests on the ability to read, write, produce or comprehend numerals (*number transcoding*; e.g., Deloche & Seron, 1987). Therefore number processing, in its fundamental form, seems intuitively linked to the ability to mentally manipulate sequences of words or symbols according to fixed transcoding or calculation rules. Hurford (1987), following Chomsky (1980), argues that "the number faculty largely emerges through the interaction of central features of the language faculty with other cognitive capacities relating to the recognition and manipulation of concrete objects and collections . . . . It is therefore not necessary to postulate an autonomous 'faculty of number' as a separate module of mind" (p. 3). This view of what constitutes the core of numerical ability, although it is rarely articulated as clearly, is widely spread, and it has been largely successful in modelling adult human arithmetical performance.

A short history of formal number notations

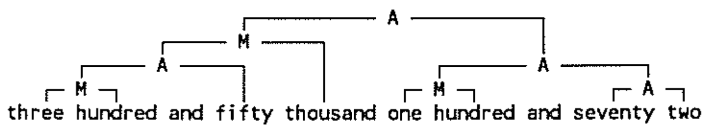
Historically, number notations have passed through several stages of increasing efficiency (Dantzig, 1967; Guitel, 1975; Ifrah, 1981; Menninger, 1969). This evolution is important to consider because it has influenced our present notational systems. The simplest numerical notations are *concrete* (Ifrah, 1981): a number of similar tokens (e.g., fingers, notches on a stick, knots on ancient *quipu* Inca strings) are put in one-to-one correspondence with the denumerated set, and the resulting pattern of tokens serves as a number word. Thus, "four" is written |||| in Egyptian hieroglyphic notation. The category of concrete notations also includes the peculiar gestural notations used in New Guinea, where numbers up to 33 are denoted by pointing to different parts of the body, or by naming the appropriate part (e.g., the word for 23 is literally "left ankle").

Written concrete notations are tedious because above some limit consecutive number words cannot be rapidly discriminated. This difficulty was obviated in many ancient cultures by grouping the marks in a recognizable pattern (e.g., 5 = ||| || in Egyptian hieroglyphs) or by inventing altogether new symbols (e.g., $5 = \Gamma$ in ancient Greek).[1] *Additive notations* later emerged as a solution to the problem of memorizing new arbitrary symbols for each number. Special symbols are attributed to some fundamental numbers (e.g., 10 and 100). The notation for any number is then obtained by juxtaposing the appropriate number of symbols of each sort (e.g., 43 is denoted "10 10 10 10 1 1 1"). Although fundamental numbers are often powers of 10, this is by no means necessary. For instance some African tribes use base 2, Sumerians used the fundamental numbers 10, 60, 100, 600, 3600 and 36,000, and Greeks had distinct symbols (letters) for numbers 1–9, 10–90 and 100–900 (Guitel, 1975; Ifrah, 1981; Menninger, 1969).

A more compact notation is achieved with *hybrid multiplicative–additive*

[1] Incidentally, such deviations from strictly concrete notations always start at 4 or 5 (Ifrah, 1981), strongly suggesting the cross-cultural invariance of the subitizing range (see below).

notations, a good example of which is provided by English verbal notation. Fundamental numbers, instead of being reproduced several times, are preceded by another number by which the fundamental number must be multiplied. For instance we say "three hundred" rather than "hundred hundred hundred". In contrast to additive notations, where word order is irrelevant, hybrid multiplicative–additive notations possess a complex syntax. The linguistic structure of hybrid numerals (Hurford, 1975, 1987; Power & Longuet–Higgins, 1978) can be described as a tree with successive embeddings of multiplication and addition. Thus, if M denotes multiplication and A addition, the English numeral for 350,172 can be decomposed in the following way:



In hybrid notations, two classes of basic number words must be distinguished: words like "three" that refer to specific numerical quantities, and multiplier words like "hundred", that Guitel (1975) calls σ signs, which often have no intrinsic numerical value and serve as grammatical markers in the word string. In the simplest base-10 multiplicative–additive notation, exemplified by Japanese Kanji notation, the lexicon is limited to the *ones* number words (one through nine) and the multiplier words ten, hundred, thousand, etc. For instance, 12 is written as "ten two", and 27 as "two ten seven". In the more complex English system, addition of ten and multiplication by ten are not indicated by a multiplier word but through morphological markers (-teen or -ty). As a result, the English lexicon is augmented with two number words classes, *teens* words (ten..nineteen) and *tens* words (twenty..ninety).

Despite their ingenuity, hybrid notations limit the range of denoted numbers, do not provide a compact code, and do not permit easy calculation. These defects vanish in written *positional notations*, exemplified by arabic notation. The lexicon is reduced to a small set of symbols (digits), which denote the integers that are smaller than the base. The position of each digit in the numeral determines the power of the base by which it must be multiplied (e.g., $321 = 3 \times 100 + 2 \times 10 + 1$). If only the digits 1–9 are defined, then this system is ambiguous. For instance $3 \times 100 + 1$ and $3 \times 10 + 1$ are denoted by the very similar strings "3 1" and "31". Even worse, the numbers 3, 3×10, and 3×100 are coded by the same string "3". Such ambiguities actually exist in some written notations, for instance cuneiform script (Ifrah, 1981; Menninger, 1969). To prevent them, the special symbol 0 was invented to explicitly indicate the absence of a given power of the base in the decomposition of a number. Digit zero, before acquiring a meaning of its own, worked only as a syntactic device.

The mental representation of numerical notations

Every English literate can produce and understand numerals in at least two numerical notations: arabic and verbal. How are the corresponding transcoding rules mentally represented? The study of brain-lesioned patients with "aphasic acalculia" – deficits in number reading, writing, auditory comprehension or verbal production – has helped to answer this question (McCloskey, this issue). The processing of arabic numerals, for instance, can be dissociated neuropsychologically from the processing of verbal numerals (McCloskey & Caramazza, 1987; McCloskey, this issue). Within each notation system, patients can be found with intact number production but impaired number comprehension, or the converse (e.g., Benson & Denckla, 1969; McCloskey, Sokol, & Goodman, 1986; McCloskey & Caramazza, 1987; McCloskey, Sokol, Goodman-Schulman, & Caramazza, 1990; Noel & Seron, 1992).

The neuropsychological approach has culminated in identifying dissociations of lexical and syntactic number transcoding processes. The two classes of impairments yield different error patterns in reading or in writing numerals. Some patients make digit or word substitution errors (e.g., they read 450 as "*three* hundred fifty"), but otherwise make no errors in the structure of the word sequence. This qualifies as lexical impairment with preserved syntax. Detailed studies of the breakdown of the mental lexicon for numerals have revealed a micro-organization in separate stacks for ones, teens, tens, and perhaps also multiplier words (Deloche & Seron, 1982a, 1982b, 1984; McCloskey et al., 1986; see McCloskey, this issue), thus confirming the linguistic analysis of numerical notations (Hurford, 1975; Power & Longuet-Higgins, 1978).

Other patients process individual digits or number words correctly, but fail to combine them. For instance, French aphasics may transcode the verbal numeral "sept cent mille" (700,000) into the arabic numeral "1,107" (mille cent sept), and therefore ignore the multiplicative structure implied by word order (Deloche & Seron, 1982a, 1982b, 1987; Seron & Deloche, 1983). Such a pattern suggests an impairment of number syntax. Elaborate analyses of patients' transcoding errors have permitted the development of precise models of normal number processing. For instance, McCloskey et al. (1986) have modelled the successive steps intervening in the production of a numeral in verbal notation (for review see McCloskey, this issue).

The lexical versus syntactic distinction in number transcoding is corroborated by studies of number acquisition in children. Children acquiring the sequence of number words make two types of errors. First, they sometimes use one number word in place of another, for instance, invariably counting "one, two, six" (Gelman & Gallistel, 1978); this qualifies as a lexical error. Second, they may invent number words such as twenty-ten, twenty-eleven, etc. (Fuson, 1982, 1988; Seron & Deloche, 1987; Siegler & Robinson, 1982). This represents an over-generalization of the inferred rules of number syntax (see Power & Longuet-

Higgins, 1978, for a computational model of number grammar induction). The rate and typology of such syntactic errors depends on the complexity of the acquired number notation, e.g., Chinese, English, or Korean (Miller & Stigler, 1987; Song & Ginsburg, 1988).

Formal calculation processes

With the mastery of a positional system such as arabic notation comes the ability to calculate, that is, to predict by symbolic manipulation the result of a physical regrouping or partitioning act without having to execute it. Over recent years, adult and infant performance in addition and multiplication, and to a lesser extent in subtraction and division, has been the focus of extensive research. Results in this area, which is referred to as *cognitive arithmetic*, are thoroughly reviewed by Ashcraft (this issue). For the sake of completeness of the present introduction, the main findings are summarized below.

Single-digit operations

The fundamental result of cognitive arithmetics in normal adults is the *problem size effect*. The time to solve a single-digit addition or multiplication problem such as $2+3$ or 4×7 increases with the size of the operands (e.g., Parkman, 1972; Parkman & Groen, 1971; Svenson, 1975; for review see Ashcraft, this issue). For instance, computing 8×9 may take 200 ms more than computing 2×2. The increase is generally non-linear: calculation time correlates well with the product of the operands, or with the square of their sum. A notable exception is the case of ties (e.g., $2+2$, 4×4), for which response time is constant or increases only moderately with operand size (Miller, Perlmutter, & Keating, 1984; Parkman, 1972).

Ashcraft (this issue) describes in great detail and in a historical perspective the variety of models that have been proposed for these results. Although several models remain in competition, all now share the notion that in the adult, arithmetical facts such as $2 \times 2 = 4$ are memorized and retrieved from a stored mental network or lexicon. The problem size and tie effects are viewed as reflecting the duration and difficulty of memory retrieval. These effects are comparable to frequency effects in lexical access. According to Ashcraft (1987), they faithfully reflect the frequency with which arithmetical facts are acquired and practised (see Gallistel & Gelman, this issue, for contrasting views).

Many results in cognitive arithmetics are nicely embraced by the analogy of stored addition and multiplication tables with a lexicon. Spreading activation among related facts can account for the difficulty of rejecting problems like

$5 \times 7 = 30$, where the proposed result falls in the same row or column as the correct result (Stazyk, Ashcraft, & Hamann, 1982). Calculation errors also tend to follow a similar pattern (Campbell & Graham, 1985; Ashcraft, this issue). As in word recognition, repetition priming obtains in arithmetical fact retrieval: a problem like $7 + 4 = 11$ is classified faster the second time it is presented (Ashcraft & Battaglia, 1978). Error priming, whereby an erroneous response to a given arithmetic problem is selectively enhanced by the prior presentation of a problem for which this response was correct, also obtains (Campbell, 1987; Campbell & Clark, 1989). Thus, after processing 4×6, the probability of erroneously responding 24 to 3×7 is increased. Such priming effects may last for one minute or more (Campbell & Clark, 1989), suggesting that stored arithmetical facts behave like logogens with a fluctuating threshold of activation (Morton, 1970).

Finally, evidence suggests, if not definitively, that the arithmetical store is accessed automatically. Zbrodoff and Logan (1986) found slow verification responses to problems like $3 + 4 = 12$, in which the proposed result is actually the result of the wrong operation (see also Winkelman & Schmidt, 1974). This suggests that both multiplication and addition are initiated irrepressively from the presentation of an arithmetical problem. LeFevre, Bisanz, and Mrkonjic (1988) found that the mere presentation of two digits like "4 2" in a memory task yielded an automatic activation of the addition result 6: subjects took more time to verify that a probe digit did not belong to the previous set when this probe was equal to the sum of the digits (6) than when it was another unrelated digit (e.g., 3).

The study of neuropsychological deficits in arithmetical fact retrieval has basically confirmed the picture obtained from normal subjects (McCloskey, Aliminosa, & Sokol, 1992; McCloskey, this issue). Brain damage can selectively impair memory for addition and multiplication tables. Disruption is often scattered, affecting, for instance, the retrieval of 8×8 but not of 9×8 or 8×9. This suggests that these facts are mentally represented independently from one another. An exception to this pattern is the case of multiplications by 0 and by 1, and perhaps also additions of 0 (McCloskey, this issue; McCloskey et al., 1992). Many patients, when they fail, for instance, on problem 0×1, also tend to fail on other related problems 0×2, 0×3, etc. In case of spontaneous recuperation, such problems also tend to be recovered simultaneously in time. Neuropsychological data therefore support the proposition made by several authors (e.g., Baroody, 1983, 1984a; Parkman, 1972) that part of our arithmetical knowledge is stored in rules of the form $0 \times N = 0$, $1 \times N = N$ or $N + 0 = N$.

Multi-digit calculation procedures

Introspectively, calculation with multi-digit numerals involves the sequential combination of elementary arithmetical operations using a specific algorithm learned at school. Psychological research with normal adults essentially confirms this model (Ashcraft & Stazyk, 1981; Geary, Widaman, & Little, 1986; Widaman, Geary, Cormier, & Little, 1989; Timmers & Claeys, 1990). The time to complete a multi-digit operation such as 24 + 37 is accurately predicted by the retrieval time of the elementary facts (4 + 7 and 2 + 3), plus some precise latency for the encoding of the operands and for possible carry operations. As might be expected, the calculation of additions generally proceeds columnwise from right to left. When an operation is verified, the calculation stops as soon as an error is detected in the proposed result. Encoding, carrying and sequencing operations seem common to addition and multiplication (Geary et al., 1986).

The study of calculation error in brain-lesioned patients confirms the existence of dissociable processes for the retrieval of arithmetical facts and for their sequencing. It is frequent for a patient to retain the ability to perform single-digit operations while making gross errors in multi-digit operations, or the converse (e.g., Caramazza & McCloskey, 1987; McCloskey, this issue). Even normal subjects, especially children, often make systematic calculation errors such as failing to carry properly in multi-digit subtraction problems. These systematic errors have been characterized as repairs to a faulty mental calculation algorithm (van Lehn, 1990).

Alternative calculation strategies in childhood

A fundamental question in cognitive arithmetics concerns the manner in which children converge to adult calculation abilities (Ashcraft, 1982, 1987, this issue; Siegler & Shrager, 1984). The majority of investigators have concentrated on the operation of addition, which shows a clear developmental trend from early counting-based strategies to adult memory retrieval. In young children, a frequent strategy is the *counting on* procedure, in which children start with the larger of the operands and count upward as many times as is required by the smaller of the operands (Groen & Parkman, 1972; Svenson, 1975). Thus 4 + 3 is calculated by counting 4, 5, 6, 7. This is also called the *min* strategy because calculation time is accurately predicted by the minimum of the two addition operands. It is closely related to the alternative *counting all* strategy, in which the child counts from one up the number of times indicated by the first and then the second operand (e.g., for 4 + 3: 1, 2, 3, $\underline{4}$, 5, 6, $\underline{7}$; Baroody & Ginsburg, 1986; Fuson, 1982).

Several other strategies are available to the child: guessing, decomposing the problem (e.g., 4 + 8 = (4 + 6) + 2), retrieving the answer from memory, etc. It is not true, as was initially thought, that the use of such strategies follows a strict developmental sequence, or that children use only one strategy at any given time in development. Rather, individual children typically switch between strategies from trial to trial (Siegler, 1987a), and which strategy is selected depends on the reliability and speed of the available strategies, as measured over previous calculation trials (Siegler & Shrager, 1984).

During the development of simple arithmetical abilities, memory retrieval progressively wins over other calculation processes. Theoretical acquisition models, reviewed by Ashcraft (this issue), attribute the speed of such memory strengthening either to the order and frequency of presentation of arithmetical facts (Ashcraft, 1987), or to interference from erroneous results that compete with the correct result for memorization (Siegler & Shrager, 1984). The acquisition of subtraction and multiplication facts has been similarly studied and modelled (Ashcraft, 1982; Graham, 1987; Miller & Paredes, 1990; Siegler 1987b, 1988). Finally the acquisition of algorithms for multi-digit operations, a case of induction of a procedure from examples, has been modelled by van Lehn (1986, 1990).

Conclusion

The language faculty has endowed humans with the ability to develop number notations especially tailored to their calculation and communication needs. Most of adult number processing relies heavily on these notational devices, thereby explaining the predictive power of models assuming a mental "algebra" of symbol manipulations. Mental calculation errors can even be described as "bugs" within a mental calculation program (van Lehn, 1986, 1990). Nevertheless, in the rest of this paper, the view that number processing reduces to linguistic or symbolic processing is challenged as a general model for animal, infant or even adult numerical abilities. It is argued below that identification of numerical competence with calculation and transcoding neglects two fundamental domains, quantification and approximation, that do not seem to be based on symbol manipulation.

II. QUANTIFICATION AND THE NATURE OF PREVERBAL NUMERICAL KNOWLEDGE

At the most concrete level, number is a property of sets of objects in the external world, which must be recognized and mentally represented before any form of numerical cognition can develop. To escape the ambiguity of the word "number", the term *numerosity* is used to refer specifically to a measurable numerical quantity, and the neologism *numeron* denotes a mental representative of numerosity (Gelman & Gallistel, 1978). *Quantification* consists in grasping the numerosity of a perceived set and accessing the corresponding (possibly approximate) mental token or numeron. Three quantification processes have been postulated: counting, subitizing, and estimation (Klahr, 1973; Klahr & Wallace, 1973). We shall consider these three processes in turn.

Counting

Gelman and Gallistel (1978) have proposed a now universally adopted definition of counting in terms of five principles:

(1) *One to one correspondence:* each element of the counted set must map onto one and only one numeron.
(2) *Stable order:* the numerons must be ordered and mapped in a reproducible sequence onto the items to be counted.
(3) *Cardinality:* the last numeron used during a count represents a property of the entire set (its cardinality or numerosity).
(4) *Abstraction:* counting applies to any collection of entities (all sorts of physical objects, possibly forming a heterogeneous set, as well as purely mental constructs).
(5) *Order irrelevance:* the order in which different elements of the counted set are mapped onto numerons is irrelevant to the counting process.

Gelman and Gallistel's (1978) definition of counting puts no constraints on the nature of the numerons, except that they must be reproducibly ordered. The definition allows for the use of an idiosyncratic list of number words: a child who reproducibly counts "one two six . . ." may nevertheless master the principles of counting. It does not imply either that the numerons are words of the language: hand or body gestures, as used by African and New-Guinean tribes, may support counting as well. According to Gallistel and Gelman (this issue), competence for counting and competence for language are largely distinct. Counting is therefore accessible in principle to non-linguistic animals as well as to prelinguistic human infants.

In fact there is now a wealth of evidence for elementary number processing in animals, which has been reviewed elsewhere (Gallistel & Gelman, this issue; Davis & Pérusse, 1988; Gallistel, 1990). For instance, Matsuzawa (1985) trained a chimpanzee to press a key with the appropriate arabic digit in response to sets of 1–6 objects. Meck and Church (1983) conditioned a rat to press one lever in response to a sequence of 2 beeps, and another lever in response to a sequence of 8 beeps. Even when duration was confounded with numerosity during the training phase, the rats gave evidence of subsequent generalization on the basis of numerosity alone.

Animal numerical discrimination is not limited to small numerosities, but may extend, for instance, to sequences of 50 events (Rilling & McDiarmid, 1965). However, the variance in the animal's representation of numerosity apparently increases in proportion with the input numerosity. This "scalar property" suggests the use of a noisy quantification procedure. In Meck and Church's (1983) model of animal counting, numbers are represented internally by the continuous states of an analogue accumulator. For each counted item, a more-or-less fixed quantity is added to the accumulator. The final state of the accumulator therefore correlates well with numerosity, although it may not be a fully precise representation of it. At odds with the human slow and self-monitored verbal form of counting, this model endows the animal with a fast, mechanical, but intrinsically approximate counting device. Gallistel and Gelman (this issue) suggest that human infants are equipped with a similar preverbal counting mechanism that may "bootstrap" the acquisition of the verbal number system.

When do children understand counting?

While Gelman and Gallistel's (1978) definition widens the range of counting behaviours, for instance allowing for non-verbal counting, it also puts stronger demands on the identification of genuine counting in children. It is not sufficient for a child to recite the adequate series of number words in one-to-one correspondence with the elements of the counted set. Cardinal, abstraction and order-irrelevance principles must also be satisfied. Concerning the emergence of these principles in childhood, two opposing theoretical views have been proposed. Gelman and Gallistel's (1978) *principles-first* theory states that principles are innate and guide the acquisition of counting procedures. This view revived the old competence/performance distinction. Thus, Greeno, Riley, and Gelman (1984) favoured a three-tiered model, with conceptual competence (abstract constraints or principles) and utilization competence (understanding of task demands) predating the generation of adequate behavioural procedures. In sharp contrast, the *principles-after* theory (e.g., Briars & Siegler, 1984; Fuson, 1988; Fuson & Hall, 1983) states that counting principles are progressively abstracted, in a Piagetian manner, after repeated practice with imitation-derived rote counting procedures.

The assessment of a putative principled competence behind simple counting performance has resulted, in the last ten years, in a very rich body of research. Attention has been drawn not so much to what children can do, but to what they understand about what they do. Can they monitor their counting errors, or detect errors in a puppet's counting? Can they invent adequate procedures for unusual counting situations, such as counting a circular display?

Gelman and Meck (1983) assessed understanding of the one–one and order-irrelevance principles by having children discriminate correct versus wrong counts by a puppet. Of interest were trials in which the puppet counted all the items correctly, but in an unconventional order instead of the standard left-to-right order. Despite never having seen such counting before, most children classified it as correct.[2] Gelman and Gallistel (1978) also showed that young children readily accepted to start their count with any given item and not necessarily the one

[2]Briars and Siegler (1984) failed to replicate this result. However, Gelman and Meck (1986) gave a convincing explanation for this discrepancy: Briars and Siegler's children were led to judge the conventionality of the count rather than its correctness.

furthest left.[3] Current evidence therefore suggests an early availability of the order-irrelevance principle.

The situation seems different for the cardinal principle. Young children often repeat the last numeron in a count (Gelman & Gallistel, 1978), and they are able to detect when a puppet fails to do so (Gelman, Meck, & Merkin, 1986). However, such last-word responding may be a rote procedure acquired by imitation. Young children fail to relate "how many" questions to previous counting (Fuson & Hall, 1983), and they do not spontaneously count when asked to give a specified number of items (Schaeffer, Eggleston, & Scott, 1974; Wynn, 1990). In such tasks, appropriate understanding of cardinality seems to emerge at only $3\frac{1}{2}$ years of age (Wynn, 1990).

Finally, as regards the abstraction principle, young children readily count heterogeneous sets of items, comprising, for instance, animate as well as inanimate objects (e.g., Fuson, Pergament, & Lyons, 1985; Gelman & Tucker, 1975). They can even count actions or sounds, although somewhat less accurately (Wynn, 1990). However, Shipley and Shepperson (1990) showed that children are extremely reluctant to count as only one item an object that is broken in two parts. Children also fail at counting kinds (how many kinds of animals?) or properties (how many colours?) when each is represented more than once. For instance, when presented with three dogs and one cat, they will say that there are four, not two, kinds of animals. The abstraction principle may therefore not be fully operative initially, and counting seems strongly biased towards discrete physical objects (Shipley & Shepperson, 1990).

Is subitizing different from counting?

Animal evidence suggests that counting does not have to be verbal. Surprisingly even in human adults, the identification of verbal counting sometimes proves elusive. In experiments with timed numerosity judgments, adult subjects are asked to determine, as fast and accurately as possible, how many items are presented in a display. Simple counting would appear to predict that response time should increase linearly with the numerosity of the display. However, such a pattern is found only over a limited range of numerosities. For instance, Mandler and Shebo (1982) found that with a 200-ms presentation time, numerosity judgment latencies increased linearly by about 300 ms per item only over the range 4–6. For numerosities 1–3, response times (RTs) were fast and increased only moderately with the number of items. And for numerosities larger than 7,

[3]Baroody (1984b) questioned the implications of this study, showing that children believe that counting in two different orders may yield two different results. However, according to Gelman, Meck, and Merkin (1986), Baroody's children gave two different numbers on the two different counts because they interpreted the experimenter's query as meaning that their initial response was false.

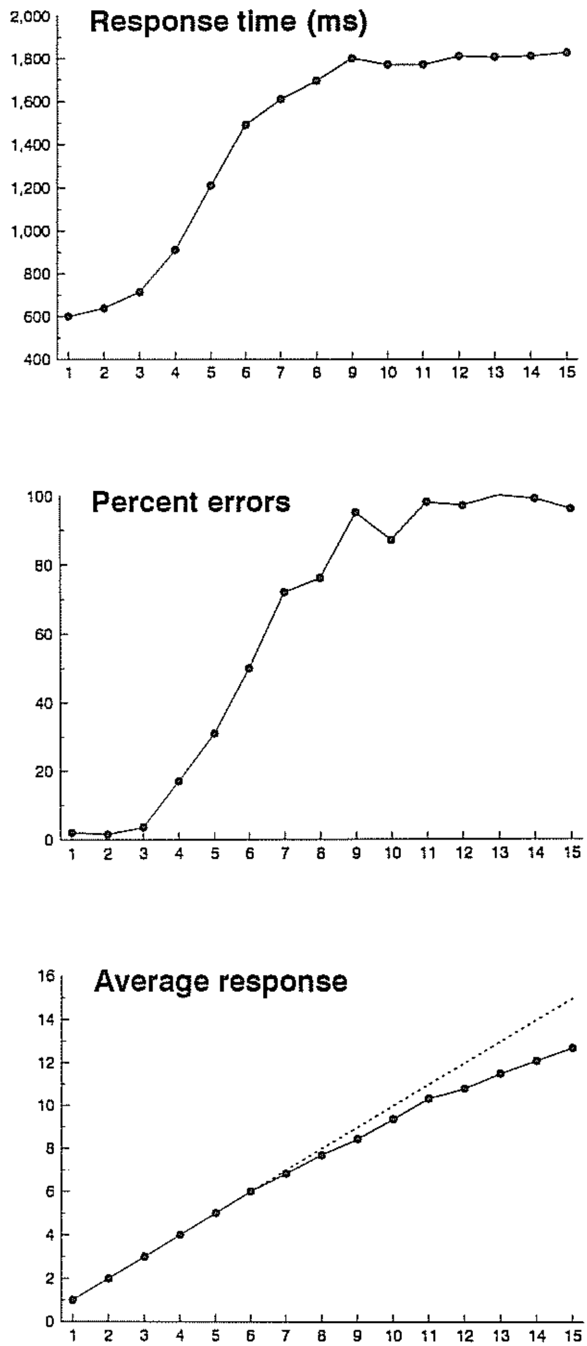


Figure 1. *Performance in timed quantification of visual displays presented for 200 ms (adapted from Mandler & Shebo, 1982). Response time, error rate and average response are plotted as a function of the actual numerosity of the display.*

RTs were approximately constant, but accuracy dropped severely (Figure 1). This pattern suggested that counting was used only in the range 4–6. The term "subitizing"[4] was coined for the process responsible for fast responses to small numerosities (Kaufman, Lord, Reese, & Volkmann, 1949), and the term "estimation" for the less accurate process used preferentially with large numerosities.

The existence of subitizing and estimation processes distinct from counting remains a highly controversial issue. Subitizing remains defined merely as "a somewhat mysterious but very rapid and accurate 'perceptual' method" (Beckwith & Restle, 1966, p. 443), and is therefore criticized for lacking a precise theoretical characterization. Gallistel and Gelman (1991, this issue) defend the radical view that subitizing is nothing more than counting at a fast rate using non-verbal numerons. Dwelling on Meck and Church's (1983) model for animal counting, they propose that human infants and adults possess a similar fast but inaccurate counting method. Because the variability of the count increases in proportion with numerosity, accurate naming is feasible only over a small range of numerosities, say from 1 to 4. Even over this "subitizing" range, RT is expected to increase with numerosity, albeit at a low rate. Such an increase from 1 to 2 and from 2 to 3 is indeed experimentally observed (e.g., Chi & Klahr, 1975; Mandler & Shebo, 1982).[5]

Subitizing by recognition of canonical configurations

Mandler and Shebo (1982) have outlined an alternative subitizing model based on the recognition of canonical configurations of visual items. In visual displays with a constant but small number of items, the disposition of objects necessarily forms invariant or canonical spatial configurations which may be recognized in parallel: one = a dot, two = a line, three = a triangle. Our visual system may recognize "threeness" in a triangular configuration, whatever the exact nature and arrangement of the constituent objects, just as it can recognize a cow regardless of viewpoint, size, colour, etc.

[4]Confusingly, the meaning of the word "subitizing" has shifted over the years. Initial studies of fast quantification (e.g., Taves, 1941; Kaufman, Lord, Reese & Volkmann, 1949) suggested the existence of two mechanisms: one for numerosities up to 7, the other for larger numerosities. The first mechanism was named subitizing. Later, however, a second distinction was introduced. Fast and almost flat RTs were obtained over ranges 1–3 or 1–4; for larger numerosities RTs increased linearly at a much steeper rate, suggesting the use of counting (e.g., Chi & Klahr, 1975; Mandler & Shebo, 1982). The previously reported limit of 6 or 7 on accurate quantification was not always replicable and varied with the duration of presentation of the displays (Averbach, 1963). It was thus thought merely to reflect the number of "discrete events [that] can be held in consciousness and counted" (Mandler & Shebo, 1982, p. 18; Miller, 1956). Thereafter, the term subitizing was used, as in the present article, to refer only to the fast process operating over the range 1–3 or 1–4.

[5]Sagi and Julesz (1985) reported a high and constant quantification performance in tachiscopic presentation of sets of 1–3 items, but this result failed to be replicated (Folk, Egeth, & Kwak, 1988).

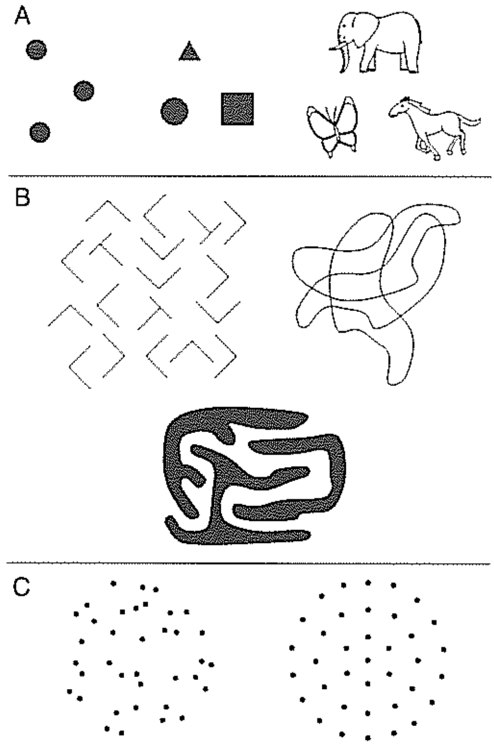


Figure 2. *Capacity and limits of the subitizing and estimation procedures. (A) Easily subitized sets. Three non-overlapping objects usually form a recognizable triangular configuration. (B) Sets for which subitizing by recognition of canonical configurations is impossible in principle. Ts do not pop out from Ls, hence their configuration cannot be recognized in parallel. Overlapping transparent shapes, as well as tangled shapes, do not occupy well-specified spatial locations which could form a recognizable configuration. (C) Sets that are preferentially quantified using an estimation procedure. Under rapid viewing conditions, random sets (left) are systematically underestimated whereas regular sets (right) are generally overestimated (this effect may not obtain under free-viewing conditions). Both sets actually contain 37 items.*

Even allowing for very powerful shape recognition processes, no visual system could possibly recognize a unique configuration for "ten-ness" that would be present in all displays with ten items. If only linear operations are permitted (image translation, rotation, projection and size deformation), then simple mathematical arguments predict that only configurations of 1, 2 or 3 objects can be recognized.[6] The special case of 4 might also be handled since it might be coded with only two canonical configurations (a square or a triangle with a dot inside).

[6]For instance, only for $n \leqslant 3$ can any 2-dimensional configuration of n points be obtained by linear deformation of a single canonical 2-dimensional figure (e.g., an equilateral triangle for $n = 3$). Similarly, only for $n \leqslant 3$ can any 2-dimensional configuration of n points be obtained by projection on the plane of a single 3-dimensional configuration of n points.

This predicted subitizing range of 3 or 4 items is of course in excellent agreement with the experimental data (e.g., Chi & Klahr, 1975; Mandler & Shebo, 1982; see Figure 1). Additionally, the cost of bringing a visual pattern in register with its canonical configuration should increase with numerosity. Less image processing is presumably needed to recognize a single dot than to recognize a triangle in a random configuration of three dots. Therefore, like preverbal counting, the model correctly predicts that naming time should increase moderately with numerosity over the range 1–3.

The canonical configuration model predicts flat naming times when the patterns presented coincide with memorized canonical configurations. Mandler and Shebo (1982) ran a condition in which all sets of three dots were arranged in the form of an equilateral triangle, all sets of four dots in the form of a square, and so on. RTs were then found to be absolutely flat over the range 1–5. Over the range 1–3, RTs to canonical displays were not significantly different from RTs to randomly arranged displays, which Mandler and Shebo took to imply that recognition of canonical configurations was indeed normally employed over this range only.

Other data confirm that subitizing taxes low-level, preattentive visual recognition processes. First, subitizing occurs only when the visual items "pop out" effortlessly from the background, but not when serial attentive processing is necessary to isolate the targets; for instance, in a situation of feature conjunctions (Trick & Pylyshyn, 1989) or when counting letters "O" among distractor letters "Q" (Trick & Pylyshyn, 1988). Second, subitizing seems to operate only above some minimal inter-item separation. Atkinson, Campbell, and Francis (1976) have shown that the numerosity limit at which quantification becomes slow and erroneous drops from 4 to 2 if the interval between dots in a linear array drops under 0.05 degrees of visual angle (still an easily resolved separation). Finally, the subitized items must occupy distinct and rapidly identifiable positions in space. Concentric rectangles, for instance, cannot be subitized (Trick & Pylyshyn, 1988; see Figure 2B).

Non-geometrical models of subitizing

Despite this experimental support, the canonical configuration model faces one severe difficulty. Fast naming times indicative of subitizing are observed even in experiments in which the existence of geometrical configurations is dubious, for instance when the items are arranged in a single line (e.g., Atkinson, Francis & Campbell, 1976). To accommodate this fact, one may extend the definition of a canonical configuration to include, for instance, "line of 3 points" as a recognizable visual object. Alternatively one may postulate that the ill-specified visual normalization processes are able to recognize a potential triangle in a line of 3 dots. Both solutions seem to beg the question of subitizing since they do not explain why such powerful object recognition is not available for numerosities higher than 4 or 5.

Some authors have suggested that the information accessed during subitizing is not geometric but more abstract. According to Trick and Pylyshyn (1988, p. 1), "subitizing and counting are side effects of the way the visual system is built: there is a parallel (preattentive) stage and a spatially serial (attentive) stage". Trick and Pylyshyn (1988, 1989, 1991) postulate a limited number of spatial indexes called FINSTs (FINgers of INSTantiation) that automatically bind to each visual object and render it available to attention-requiring visual routines (Ullman, 1984). In subitizing, subjects would simply report the number of FINSTs currently bound to visual objects. In this model, however, the 3–4 limit on subitizing is not really explained. It is supposedly determined by the number of FINSTs, which is in itself an arbitrary parameter. Furthermore, the critical stage of affectation of one and only one FINST to each object is left unspecified. Solving this complex problem by a parallel algorithm may not be feasible. But if FINST binding is mediated by a fast serial mechanism, the model becomes equivalent to Gallistel and Gelman's (1991; this issue) preverbal counting hypothesis.

Other authors have suggested that subitizing is not an independent procedure, but merely reflects the application of a general estimation process to small numerosities. In one or two seconds, adults can estimate the numerosity of a set of up to several hundreds of dots (Ginsburg, 1976, 1978; Indow & Ida, 1977; Kaufman et al., 1949; Krueger, 1972, 1982; Mandler & Shebo, 1982; Minturn & Reese, 1951; Taves, 1941). Estimation variability smoothly increases with larger numerosities (Krueger, 1982). For instance, detecting a difference of 1 between two numerosities is easier with small numerosities (e.g., 5 vs. 6) than with large ones (e.g., 8 vs. 9; see Buckley & Gillman, 1974; van Oeffelen & Vos, 1982).[7] Possibly over the range 1–4, variability in the internal representation of estimated numerosities is so low that these numerosities can easily be separated (Averbach, 1963; van Oeffelen & Vos, 1982). Thus, the "subitizing range" would simply be the range over which estimation is sufficiently precise to yield a unique candidate numeral. This range need not be constant, but may vary with the type and discriminability of the displays, shattering the hypothesis of a subitizing process specific to sets of up to 3 or 4 items.

Detailed mathematical models of numerosity estimation have been proposed (e.g., Allik & Tuulmets, 1991; van Oeffelen & Vos, 1982; Vos, van Oeffelen, Tibosch, & Allik, 1988). Numerosity may be evaluated as a simple relation

[7]Incidentally, this form of Weber's law is strikingly similar to that observed in animal "counting" experiments (Gallistel & Gelman, this issue; Meck & Church, 1983). Could it be that the human "estimation" algorithm is essentially identical to the "counting" algorithm that rats and pigeons use? If that is the case, then a consistent terminology should be adopted.

between physical quantities, for instance the product of the visual area by the density of the items. Various numerosity illusions can be explained by a misperception of the area occupied by the visual items (Allik & Tuulmets, 1991; Bevan & Turner, 1964; Frith & Frith, 1972; Vos et al., 1988). Cuneo (1982) has further proposed that children and perhaps even adults use an incorrect Area + Density rule instead of the correct Area × Density rule. This may explain the frequent undershooting observed in numerosity estimation (Ginsburg, 1976, 1978; Indow & Ida, 1977; Kaufman et al., 1949; Krueger, 1972, 1982; Mandler & Shebo, 1982; Minturn & Reese, 1951; Taves, 1941). Thus the estimation model may account for several facts outside of subitizing *per se*.

Do infants subitize?

The adult data just reviewed do not allow us to firmly evaluate the many models of subitizing that have been proposed. Infant research, however, has recently brought more light to bear on the issue. In the first year of life, well before they learn to count, infants can discriminate sets of simultaneously presented objects on the basis of numerosity (Antell & Keating, 1983; Starkey & Cooper, 1980; Strauss & Curtis, 1981; Treiber & Wilcox, 1984; van Loosbroek & Smitsman, 1990). Infant results are grossly congruent with the 3–4 limit for fast apprehension of numerosity in adults. Even 4-day-old infants discriminate 1-object versus 2-object displays and 2-object versus 3-object displays. Discrimination seems less replicable for displays of 3 versus 4 and 4 versus 5, and it systematically fails for 4 versus 6. In addition, 4-day-old infants discriminate bisyllabic from trisyllabic words in the auditory modality (Bijeljac-Babic, Bertoncini, & Mehler, 1991), and around 6 months of age they discriminate the sequential presentation of 2 versus 3 visual objects (Davis & Ashmead, 1991).

Recognition of canonical patterns is hardly feasible with sequential visual or auditory stimuli. The data suggest that infants use a form of approximate covert counting, as proposed by Gallistel and Gelman (1991, this issue). This conclusion is also supported by cross-modal studies: 6–8-month-old infants can detect numerical correspondences between the visual and auditory modalities (Moore, Benenson, Reznick, Peterson, & Kagan, 1987[8]; Starkey, Spelke, & Gelman, 1983, 1990). When infants hear three drum beats, they look reliably longer at a visual display with three objects than at a simultaneously presented visual display with two objects. Conversely, they preferentially look at the 2-object display when hearing two drumbeats. In another experiment, 6–9-month-old infants were initially familiarized with several displays, always comprised of either two or three

[8]Moore et al. (1987) presented their results as a failure to replicate Starkey et al.'s (1983) experiment. Starkey et al. (1990), however, reanalysed the Moore et al.'s data and convincingly argued that cross-modal matching of numerical information was present in their experiment too.

objects. Subsequently, preference was assessed for auditory sequences of two versus three drumbeats. Infants showed greater interest in the auditory sequence whose numerosity matched the numerosity that they had been presented during visual habituation (Starkey et al., 1990). The straightforward explanation of cross-modal numerosity matching, endorsed by Starkey et al. (1983, 1990), is that infants perceive the relation of one-to-one correspondence between items in the visual and auditory modalities.

Conclusion

Infant, adult and animal evidence suggests the existence of several quantification processes specific to the apprehension of numerosity. Children and adults rely mostly on verbal counting. However, small numerosities can also be rapidly subitized, and large numerosities can be approximately estimated. Despite considerable experimentation and modelling, it is still not known whether these subitizing and estimation procedures radically depart from Gelman and Gallistel's (1978) definition of counting, or whether they in fact correspond to a form of fast non-verbal counting. Animal studies support Gallistel and Gelman's (this issue) assertion that animals routinely count using approximate internal quantities for numerons.

The intuitive symbolic-processing model of human numerical cognition must therefore be supplemented with a number of dedicated non-verbal quantification processes. However, these quantification processes may only represent alternative input routes to a central symbolic number processor. The nature of human adult number representations is discussed below.

III. ADULT APPROXIMATION AND PROCESSING OF QUANTITIES

Do adults process numerals using purely syntactic devices that are blind to the quantities to which the numbers refer? Transcoding errors such as "one thousand nine hundred" written as "10009100" (Deloche & Seron, 1982a), or calculation errors such as $45 + 8 = 1213$ (Caramazza & McCloskey, 1987) or $75 - 25 = 410$ (van Lehn, 1986), certainly suggest that results obtained through erroneous symbolic processing may be accepted even when they are semantically absurd. Calculation algorithms are often "applied without awareness of their conceptual basis" (McCloskey, this issue, p. 152). Of course, symbolic calculation and transcoding procedures are designed to preserve quantities. For instance, the string of digits obtained when applying the multiplication algorithm to the numerals "54" and "67", "3618", does correspond (hopefully!) to the quantity which is the product of 54 and 67. But the mathematical principles that warrant this correspondence are not accessed during calculation.

In some cases though, adults do show a sensitivity to meaningful numerical quantities. In this section I argue that tasks such as measurement, comparison of prices, or approximate calculations, solicit an "approximation mode" in which we access and manipulate a mental model of approximate quantities similar to a mental "number line" (Dehaene, 1989; Dehaene & Cohen, 1991; Dehaene, Dupoux, & Mehler, 1990; Gallistel & Gelman, this issue; Restle, 1970). To enter this putative approximation mode, arabic and verbal numerals are first translated from their digital or verbal code into a quantity code. The input modality is then neglected, and numerical quantities are represented and processed in the same way as other physical magnitudes like size or weight (*analogue* encoding). In parallel, the same numeral may also be processed via the traditional symbolic transcoding and calculation routines.

Quantity versus arabic symbols in number comparison

Perhaps the clearest evidence for a mental representation of numbers as quantities comes from the number comparison task. Moyer and Landauer (1967) showed that the time to decide which of two numbers is the larger (or the smaller) smoothly decreases with the numerical distance between them. This *distance effect* is identical whether the comparison bears on arabic numerals or whether it bears on physical parameters such as line length, pitch, or numerosity (Buckley & Gillman, 1974; Henmon, 1906). In both cases, response time is a logarithmic function of the distance (numerical or physical) between the items, and similar anchor or congruity effects are found (Banks, Fujii, & Kayra-Stuart, 1976; Dehaene, 1989; Duncan & McFarland, 1980; Jamieson & Petrusic, 1975). Even in same–different judgments, a distance effect emerges. The time to judge that two digits are different varies with the numerical distance between them (Duncan & McFarland, 1980). This suggests that digits are not compared at a symbolic level, but are initially recoded and compared as quantities.

More recently, further evidence for access to a representation of numerical quantities was obtained in a task of comparing 2-digit numbers (Dehaene et al., 1990; Hinrichs, Yurko, & Hu, 1981). Subjects had to decide whether a given 2-digit numeral, say, 59, was larger or smaller than a standard of reference, say 65. If numerals were compared on the basis of their symbolic appearance, subjects would first compare the decades digits, and then, only if necessary, the units digits. Thus, it would take the same time to compare 51 and 59 to 65: the decades 5 and 6 would be compared, and the units would not be taken into consideration. On the other hand, if the numerals were first converted into a quantity code, then the quantity codes for 51 and for 59 might be expected to differ. Because of the distance effect, it would then be faster to compare 51 with 65 than to compare 59 with 65. The latter result was indeed found: units had a significant effect on comparison times, even when the decades digit sufficed to respond "smaller" or "larger" (Hinrichs et al., 1981). RT was a smooth logarithmic function of the numerical distance between the target and the standard, with little or no discontinuity at decade boundaries. Dehaene et al. (1990) performed a variety of controls and basically confirmed this result. Their data suggested that "the digital code of numbers is [first] converted into an internal magnitude code on an analogical medium termed *number line*. This encoding stage is fast and independent of which particular number is coded" (p. 638).[9]

The SNARC effect and the orientation of the number line

We have recently performed a series of experiments which confirm that arabic numerals may rapidly and automatically evoke an internal quantity code (Dehaene, Bossini, & Giraux, 1991). Subjects were asked to judge the parity (odd vs. even) of numbers from 0 to 9. The assignment of "odd" and "even" responses to response keys was varied within subjects, so that for each number subjects responded using the right-hand key in one half of the experiment, and the left-hand key in the other half. An interaction of number magnitude with response key was found. Regardless of their parity, larger numbers yielded faster responses with the right hand than with the left, and the reverse was true for small numbers (Figure 3). This was termed the SNARC effect (spatial–numerical association of response codes).

A subsequent experiment showed that the SNARC effect is governed by the *relative* magnitude of the numbers within the range of numbers tested. When we used numbers in the 0–5 range, numbers 4 and 5, which were the largest of the interval, elicited a faster response with the right hand than with the left. This pattern reversed when we used numbers in the 4–9 range, where the same numbers 4 and 5 were now the smallest of the interval. This experiment ruled out any explanation based on the absolute characteristics of the digits, for instance their visual appearance or their frequency of usage (Dehaene & Mehler, 1992). Our interpretation is that the presentation of an arabic numeral elicits an automatic activation of the appropriate relative magnitude code. This activation cannot be repressed, even though magnitude information is irrelevant to the requested task of parity judgment. Additional data suggest that this activation persists, but is slower or less automatic, with multi-digit arabic numerals and with numerals in verbal notation.

The existence of an analogue representation for numerical magnitude does not necessarily imply the existence of a first-order isomorphism between number and

[9]Digit-by-digit numerical comparison does occur with numerals composed of 3 or more digits (Hinrichs, Beric, & Mosell, 1982; Poltrock & Schwartz, 1984).

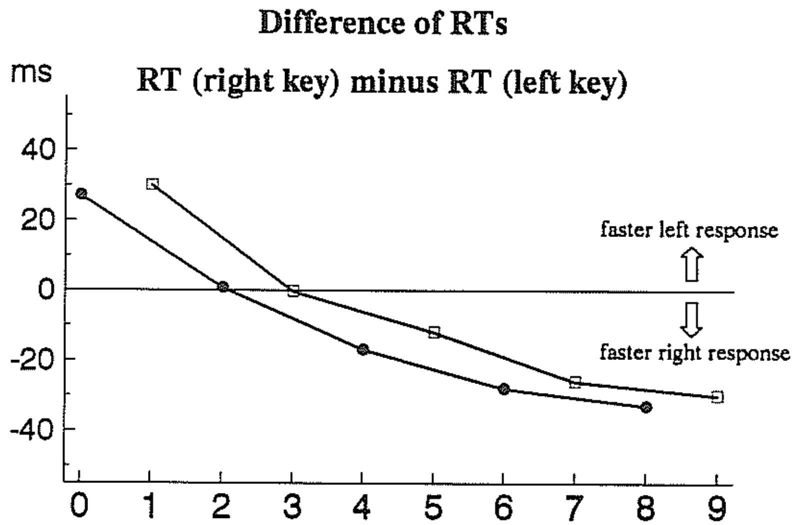


Figure 3. *The SNARC effect. In a parity judgment task, small arabic numerals are preferentially responded to with the left hand, and large arabic numerals are preferentially responded to with the right hand.*

any particular physical continuum (Shepard & Chipman, 1970). However, the SNARC effect does seem to reveal a natural mapping of the numerical continuum onto the extracorporal physical space. In subsequent experiments, we showed that the large–right association is identical in left- and right-handers, is not affected by crossing the subjects' hands, but is reversed in Iranian subjects who write from right to left (Dehaene et al., 1991). Our experiments objectify a frequently encountered intuition that the number line extends horizontally from left to right. They also mesh well with Seron et al.'s observations (this issue). Seron et al. have studied 49 normal subjects who experience peculiar visuospatial representations of number (number-forms). Their introspective number lines, even if curved or saw-toothed, frequently show a predominant left-to-right orientation. A continuum might therefore exist between ordinary subjects, who possess an (often unconscious) mental number line, the left-to-right orientation of which is identifiable only indirectly via a SNARC effect, and subjects with visual number forms, who are visually aware of their number line and show considerable elaboration around the basic left-to-right pattern.

Is the subjective scale of number compressive?

Another feature of the mental representation of numerical magnitudes, and one that is also shared by subjects with visual number forms, is the seemingly compressive character of the number line. A variety of experiments have indicated that subjective numerical magnitudes obey Weber's law: the same objective numerical difference seems subjectively smaller, the larger the numbers against which it is contrasted.[10]

One experimental technique involved asking subjects to produce random numbers in a given interval (Baird & Noma, 1975; Banks & Coleman, 1981; Banks & Hill, 1974). Subjects produced more small numbers than larger numbers, a result consistent with the hypothesis that they sampled from a flat distribution over a compressive number line. Conversely, Banks and Coleman (1981) had subjects judge how evenly and randomly a sequence of numbers sampled a given numerical interval. The best sequences were those generated by a power function with an exponent of 0.35.

In other experiments, subjects rated individual numbers using verbal category scales (very small, small, etc.; Birnbaum, 1974; Rosner, 1965), or they rated the similarity of pairs of numbers (Schneider, Parker, Ostrosky, Stein, & Kanow, 1974; Shepard, Kilpatrick, & Cunningham, 1975). Yet other techniques involved approximately bissecting numerical intervals (Attneave, 1962) or pairing numbers with physical stimuli (e.g., Rule, 1969; for review see Krueger, 1989). With the exception of one study (Rosner, 1965), all of these approaches found a compressive internal representation for numbers. Most experiments relied on off-line introspective judgments. However, the hypothesis of a Fechnerian encoding of numerical magnitudes was also found useful in modelling response times in numerical comparison tasks. Comparison times were better predicted when the distance between the two compared numbers was measured on a logarithmic rather than on a linear number line (Buckley & Gillman, 1974; Dehaene, 1989; Moyer & Landauer, 1967).

Strikingly similar results – a compressive subjective scale, with increasing similarity among larger numbers – have been found in the visual perception of numerosity (see the above discussion of numerosity estimation abilities). For instance, Buckley and Gillman (1974) tested the same subjects in two different tasks: comparison of visual numerosities versus comparison of arabic digits. The representations that they obtained using multidimensional scaling were essentially identical for both types of stimuli (both were compressive). The possibility must therefore be entertained that a single representation of approximate numerical

[10]Two conceptualizations of Weber's law are possible: (1) the number line is linear, and variability increases with numerical magnitude; or (2) the number line is a compressive logarithmic or power function, and variability is constant. The metaphor of a linear number line is sometimes preferred because it allows for an easier conceptualization of approximate addition and subtraction by juxtaposition of line segments (Restle, 1970; Gibbon & Church, 1981). However, it should be recognized that from a functional point of view the two metaphors are strictly equivalent.

quantities, obeying Weber–Fechner's law, can be accessed either via numerosity estimation, or via transcoding from arabic notation.

To some readers, the above arguments may sound absurd. Teghtsoonian and Teghtsoonian (1989) noted that if the "subjective scale of number" is a power law with exponent 0.5, as postulated by Krueger (1989), then 40 should seem only *twice* as large as 10! For Laming (1989), "the notion of a subjective scale of number is a contradiction in terms" (p. 279). But then what, if anything, did the above experiments measure? The paradoxes disappear if one accepts the existence of two largely distinct modes of number processing: one based on a symbolic code, and the other on a quantity code. Of course, numbers *qua* symbols enter into objective relations, and to talk of a subjective scale of *number* is absurd. However, if numbers can be transformed into a mental representation of quantities, and thereafter be treated just like other physical quantities, then it is no more absurd to talk of a subjective scale of *numerical magnitude* than to talk of subjective scales of weight, area, etc. The ingenuity of the above "psychophysical" experiments is to prevent, in some way or another, the use of exact digital algorithms, and therefore to probe only the analogical quantity code. An internal validation is that when this dissociation is done properly, the results converge to support a compressive internal representation of quantities.

Two dissociable number-processing pathways

Given the postulation of two largely distinct number-processing pathways – one processing numbers as symbols and the other transducing them into approximate quantities – one may hope to find their occasional neuropsychological dissociation in brain-lesioned patients. Warrington (1982) described an acalculic patient who sometimes failed to retrieve arithmetical facts in addition, subtraction and multiplication. Yet this patient always proposed numbers of plausible magnitude. For instance, when asked to solve $5+7$ he replied "13 roughly". Guttmann (1937) reported a patient (H.Ba.) who may qualify for the converse dissociation. He knew his multiplication tables and could carry out simple arithmetical calculations, but he had considerable difficulty with numerosity estimation and number knowledge.

Dehaene and Cohen (1991) have recently reported a more clearcut dissociation: the case of a severely aphasic and acalculic patient, N.A.U., who lost *all* precise knowledge of numbers and arithmetical operations, but could still translate numerals into approximate numerical quantities and process them as such. N.A.U. erred even with the simplest of calculations, producing 3 in response to $2+2$. Likewise, he could not reject $2+2=5$ as false. However, he rapidly recognized $2+2=9$ as incorrect, knowing that $2+2$ is much smaller than 9. A similar phenomenon occurred in his memory for numbers. When asked to memorize a set of 3 consecutive digits in variable order (e.g., 7 6 8), a few seconds later N.A.U. incorrectly thought that 5 was among the memorized set; however, he could still reject more distant numbers such as 2.[11]

Apparently, the only knowledge that N.A.U. could access about a number was its approximate magnitude. Consistent with the hypothesis of a recoding of numerals into quantities during larger–smaller comparison, N.A.U. was close to 100% correct in comparing 1- and 2-digit numerals, even when he could not read the stimuli aloud (his only errors occurred when comparing two close quantities, e.g., 4 and 5). But he was at chance level for judging whether a given digit was odd or even. His number knowledge was similarly affected. For instance, he would state that a dozen eggs were 6 or 10, or that a year was made up of about 350 days, again lacking the exact knowledge but preserving the correct order of magnitude. Finally N.A.U.'s reading difficulties could be interpreted in the same framework. He was almost totally unable to read letters, words or non-words. However, he managed approximately to read simple arabic and verbal numerals by counting on his fingers. For instance, he would read "3" by counting to himself while raising 3 fingers and then saying "three" out loud. Apparently, upon seeing a numeral, N.A.U. knew approximately where to stop in his recitation of the preserved canonical counting sequence.

Approximate calculations

By what mechanism could N.A.U. rapidly detect the falsehood of additions such as $2 + 2 = 9$? Counting or memory retrieval models of normal addition are quite inadequate for patient N.A.U. because his performance was not affected by the problem size effect. Contrary to normal subjects, N.A.U. did not respond faster or err less when the addition operands were small or with tie problems such as $3 + 3$ (Dehaene & Cohen, 1991). His performance seemed to depend almost exclusively on the degree of falsehood of the addition. When a grossly false result was proposed, he responded so rapidly that the use of counting seemed implausible.

The two characteristics of N.A.U.'s addition performance were, first, that he could apparently only compute an approximate result, and second, that the

[11]Interestingly, Morin, DeRosa, and Stultz (1967) report a similar effect in normal subjects. The task was to memorize a set of digits, and then to decide if a probe digit was present or absent from the memorized set. "Absent" RTs decreased with the numerical distance separating the probe digit from the set. DeRosa and Morin (1965, cited in Morin et al., 1967) also found a distance effect on "present" RTs when the memorized set was made up of consecutive digits (e.g., 3 4 5 6): numbers that were close to the set boundary (i.e., 3 and 6) were classified slower than numbers that were in the interior of the set (i.e., 4 and 5). Normal subjects seem able to represent a memorized set on their number line, and to use proximity relations on this representation to verify if a probe digit belongs to the set or not.

precision of his approximation decreased with larger numbers. For instance he would classify $43 + 21 = 69$ as correct, but not $3 + 1 = 9$. These findings are well captured by an analogical model of mental addition outlined by Restle (1970; Gallistel & Gelman, this issue). Restle proposed that addition operands are first encoded as line segments on a mental "number line". The segments are then juxtaposed mentally to obtain the result. Working with segment lengths instead of arabic digits introduces an inherent imprecision in processing, which may explain that N.A.U. could only activate a candidate set of plausible results, not the exact one. The further supposition that the segment lengths follow Weber's law may explain that the larger the operands, the coarser the precision of N.A.U.'s addition approximation.

When normal subjects verify an addition, a *split effect* is observed: the more distant the proposed result from the correct result, the faster subjects classify the addition as incorrect (Ashcraft & Battaglia, 1978; Ashcraft & Stazyk, 1981; Krueger & Hallford, 1984; Zbrodoff & Logan, 1990). This distance effect bears a definite similarity with patient N.A.U.'s response pattern. The split effect in normals might perhaps reflect a post hoc comparison of the result computed by the subject with that proposed by the experimenter. In some cases, however, responses to grossly incorrect additions are much faster than responses to correct ones. It thus seems unlikely that subjects had enough time to complete the exact calculation (Ashcraft & Stazyk, 1981; Zbrodoff & Logan, 1990). For Ashcraft and Stazyk (1981, p. 185), such evidence is "suggestive of a global evaluation process operating in parallel with [arithmetic fact] retrieval". Further research is needed to ascertain whether this global evaluation is identical to patient N.A.U.'s preserved approximate addition routine.

Availability of a magnitude representation in children and animals

I have argued that some human numerical abilities, including comparison and addition approximation, do not depend on competence for language, but require access to an analogical representation of numerical quantities. In fact this set of abilities seems to coincide exactly with the numerical competence of preverbal children and animals, thereby strengthening the conclusion that they constitute a separate preverbal system of arithmetical reasoning (Gallistel & Gelman, this issue). The ability to select the larger of two numerosities appears in children around 14 months of age (Cooper, 1984; Sophian & Adams, 1987; Strauss & Curtis, 1984), and the distance effect in numerical comparison is present from 6 years of age, the earliest age at which it has been tested (Duncan & McFarland, 1980; Sekuler & Mierkiewicz, 1977). Numerical comparison can be taught to animals (e.g., Rumbaugh, Savage-Rumbaugh, & Hegel, 1987; Thomas & Chase, 1980; Washburn & Rumbaugh, 1991; see Mitchell et al., 1985, for review), and a distance effect is also found. As in human adults, a distance effect also obtains when pigeons or rats are taught to discriminate two numerosities (Gallistel, 1990): discrimination is easier when the distance between the two numerosities is larger, thus suggesting access to an analogue representation of quantities.

Addition approximation has been much less studied, but preliminary data suggest that young children know that adding to a small set of items will augment its numerosity, even when they do not know exactly by how much (e.g., Cooper, 1984; Gelman & Tucker, 1975). In animals, the outstanding work of Rumbaugh et al. (1987) shows that chimpanzees can add two numerosities a and b, two other numerosities c and d, and choose the larger of the two quantities $a + b$ versus $c + d$ (see also Woodruff & Premack, 1981). The addition is only approximate, since animals have more difficulty when the sums fall close to each other (e.g., $1 + 4$ vs. $3 + 3$). It is striking that the abilities evident in animals and young children coincide with those that remain accessible even to deeply aphasic and acalculic human adults (Assal & Jacot-Descombes, 1984; Barbizet, Bindefeld, Moaty, & Le Goff, 1967; Dehaene & Cohen, 1991; Warrington, 1982). This supports the ontogenetic and phylogenetic precedence and modularity of numerical approximation faculties.

IV. GENERAL ARCHITECTURES FOR NUMBER PROCESSING

Three domains of numerical competence have been described: transcoding/calculating, quantification, and approximation. Each of these domains includes several specific subprocesses such as multiplication, number comparison, or subitizing, which in some cases have been shown to dissociate in brain-lesioned patients. A primary issue therefore concerns the general architecture in which these subsystems are integrated. Although this problem has rarely been considered in full generality, there have been several proposals. I shall briefly review them, and then attempt to sketch a synthesis.

Three theoretical views on the human number processing system

McCloskey (this issue) presents an integrated view of the interrelations between the various modules for number production, comprehension, and calculation. His model assumes that all numerical inputs are initially translated, via notation-specific comprehension modules, into an amodal abstract representation of numbers. Conversely, number production involves a translation from the abstract internal representation to the desired output notation via notation-specific production modules. Finally, mental calculations are performed on the amodal representation, never directly on numerals in arabic or verbal notation (see Figure

4). The hypothesis that the amodal representation is an obligatory bottleneck in number processing yields strong predictions. For instance, if, on the basis of reading errors, a deficit in arabic numeral comprehension is identified in a

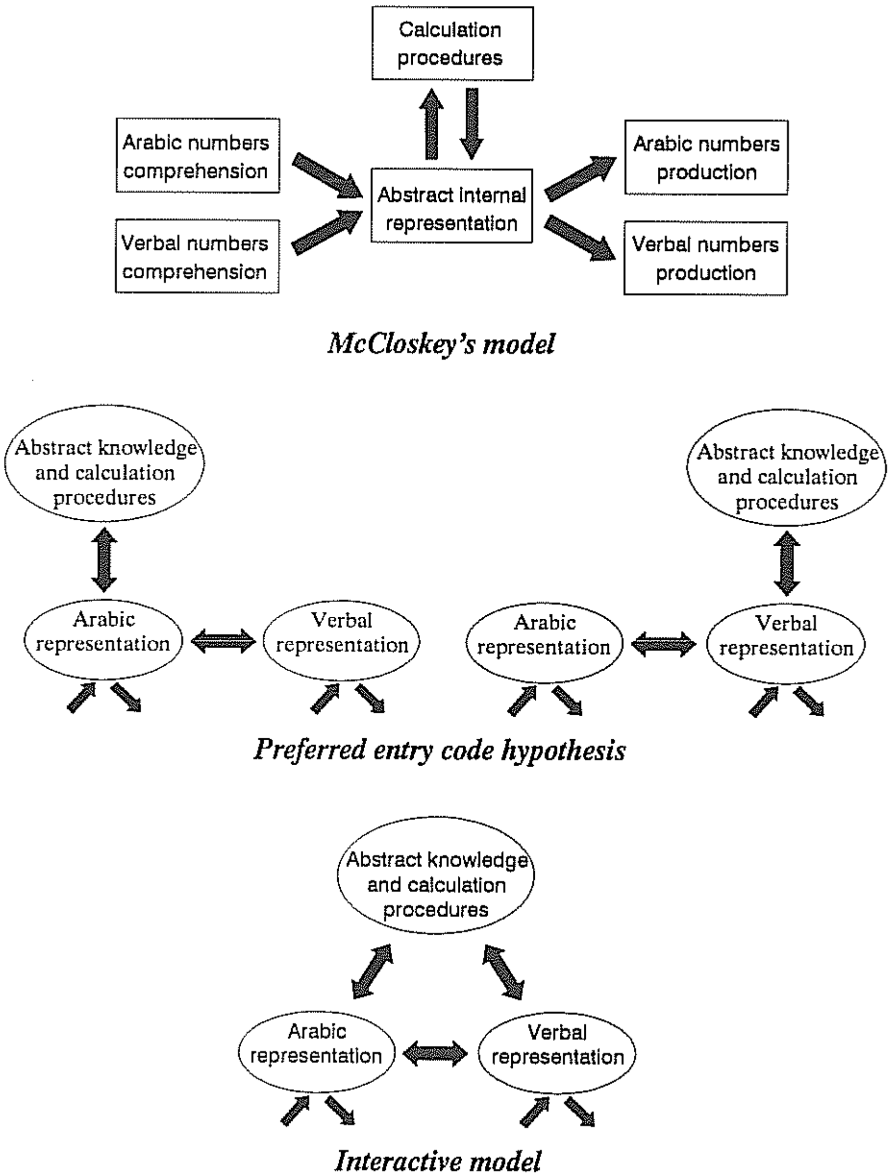


Figure 4. *Three schematic models for the architecture of the human number-processing system.*

brain-lesioned patient, one may predict that the deficit will extend to any task involving arabic inputs (e.g., numerical comparison, mental calculation). So far, this model has been extremely successful in classifying patients and in predicting their performance.

Several models have explicitly rejected McCloskey's hypothesis of a pivotal abstract number representation (Figure 4). Deloche and Seron (1982a, 1982b, 1987) have repeatedly argued for the possibility of *asemantic transcoding*, that is, direct translation between arabic and verbal notations without going through an intermediate semantic representation. Recently Noel and Seron (1992) have offered a *preferred entry code hypothesis* according to which abstract knowledge and calculation procedures are accessed via a unique notation, to which all the numerals are initially transcoded. Noel and Seron postulate that their patient NR transcodes all numerals into a verbal code through which he then accesses his number-related knowledge and a representation of quantity. The hypothesis of a verbal pivotal representation for numbers is also supported by the anecdotal observation that bilinguals prefer to perform calculations in the language in which they acquired and practised arithmetic facts (Kolers, 1968).

Conversely, to account for the performance of their patient YM, Cohen and Dehaene (1991) postulate the existence of a visual workbench or visual number form encoding numbers in arabic notation, which constitutes "an input stage common to any task involving number manipulations, including reading, magnitude comparison, calculation, etc." (p. 54). According to this view, the representation used in calculation is not abstract, but is similar to a visual image of the arrangement of the processed digits (Hayes, 1973; see also the patient described in Weddell & Davidoff, 1991).

Noel and Seron (1992) note that, just as prodigious calculators can be divided into auditory and visual types (Binet, 1894/1981), the preferred entry code may vary from individual to individual. Idiosyncratic variability would explain the elusiveness and the controversial nature of the number representation debate. A last possibility, termed the *interactive model* in Figure 4, is that all numerical codes are interconnected and that number knowledge can be accessed through any of these codes. Campbell and Clark (1988, 1992) have proposed an *encoding complex* model according to which numbers evoke "an integrated network of format-specific number codes and processes that collectively mediate number comprehension, calculation, and production, without the assumption of central abstract representation" (p. 204). Arithmetic operations such as additions or number comparisons may then be performed qualitatively differently depending on the input format of the operands (Besner & Coltheart, 1979; Gonzalez & Kolers, 1982, 1987; Takahashi & Green, 1983). The evidence supporting the interactive view is, however, still inconclusive (for discussion see McCloskey, this issue).

A triple-code model for numerical cognition

In the above models, adult transcoding and calculation tasks are considered the primary source of data. Quantification, number comparison or approximation tasks are generally not modelled in any detail. As a result the models excel in describing the syntactic processes for manipulating number notations, but do not begin to describe the semantics of number concepts. For instance, the box labelled "semantic representation" in McCloskey et al.'s (1986) model actually encodes numbers in a format very similar to arabic notation. Campbell and Clark's (1988, 1992) notion of a widespread activation of number facts in various codes seems better able to capture the richness of information that numbers can evoke (quantity, parity, etc.). However, McCloskey (this issue) rightfully emphasizes the underspecification of this idea.

The remainder of this paper, necessarily more speculative that the previous sections, is dedicated to a presentation of my own view of the human number-processing architecture. The proposed *triple-code model* is sketched in Figure 5. This model can be viewed as an attempt to reconcile Campbell and Clark's multiple-codes hypothesis with a rigorous information-processing model. It is based on two premises:

Premise 1: numbers may be represented mentally in three different codes. In the auditory verbal code or *auditory verbal word frame*, which is created and manipulated using general-purpose language modules, an analogue of a word sequence (e.g., /six//hundred/) is mentally manipulated. In the visual arabic code or *visual arabic number form*, numbers are manipulated in arabic format on a spatially extended representational medium (Hayes, 1973). In the *analogue magnitude code*, numerical quantities are represented as inherently variable distributions of activation over an oriented analogical number line obeying Weber–Fechner's law (Restle, 1970; Dehaene, 1989; Dehaene et al., 1990).

Each representation is directly interfaced by notation-specific input–output procedures similar to those present in McCloskey's model. An arabic numeral-reading procedure categorizes strings of digits for input into the visual arabic representation (Cohen & Dehaene, 1991). Conversely, an arabic numeral-writing procedure converts the internal arabic code into a motor program of writing gestures. Similar auditory input, spoken output, written input and written output procedures interface with the auditory verbal representation.[12] These procedures are not specific to numbers and also take part in the production and comprehen-

[12]An orthographic word frame might be involved in the manipulation of numerals in the written verbal format. For simplicity, however, I shall assume that written verbal numerals are parasitic on the auditory verbal system and have no direct transcoding links to other number representations.

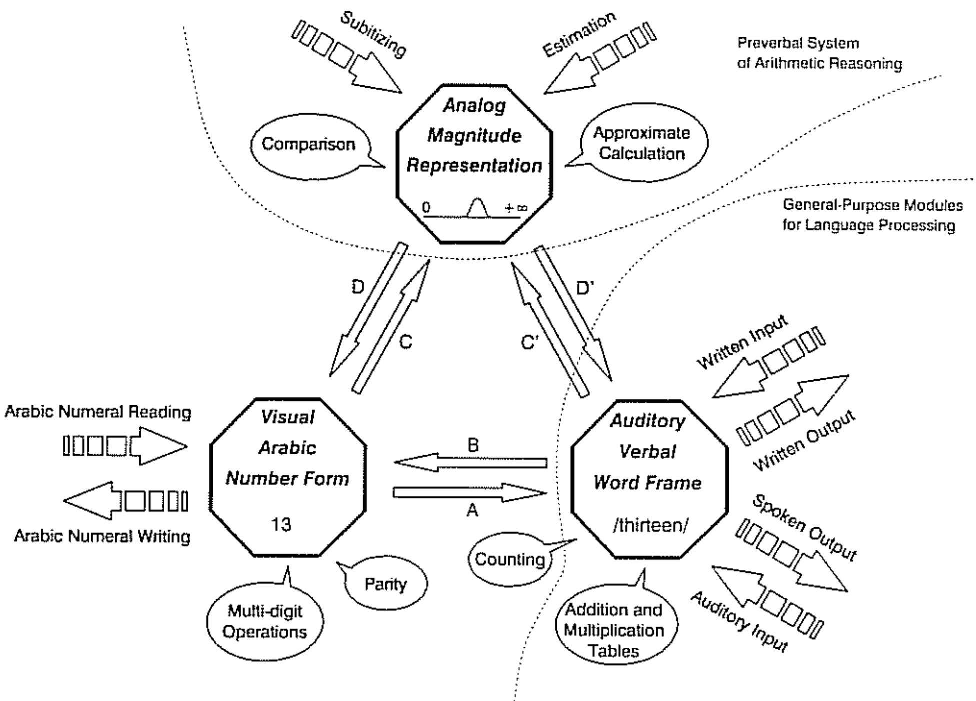


Figure 5. *Schematic representation of the proposed triple-code model. The three cardinal representations are depicted as octogons. Large arrows indicate input-output processes, thin arrows internal translation processes, and flashes operations specific to each representation. The diagram only sketches the modes of connections between various processes, several of which could be further analysed into subcomponents (e.g., counting, arabic-to-verbal translation).*

sion of oral and written language. Finally and more speculatively, the analogue magnitude representation is also assumed to receive direct input from dedicated visual numerosity estimation and subitizing procedures. This hypothesis is consistent with data suggesting similar Fechnerian psychophysical functions for visually perceived numerosity and for numerical quantity as conveyed by arabic numerals (see section III).

Communication between the three cardinal representations is achieved by dedicated translation paths marked by letters A, B, C, D, C′ and D′ in Figure 5. The arabic-to-verbal translation path (A) constructs the word sequence corresponding to a given arabic numeral. Despite its representation as a simple arrow, this is actually a complex process which involves separate steps of syntactic composition and lexical retrieval (Deloche & Seron, 1987; McCloskey et al., 1986). The reader is referred to Cohen and Dehaene (1991) for a detailed model of this asemantic arabic-to-verbal translation procedure, compatible with the current neuropsychological literature.[13] The converse verbal-to-arabic translation path (B) has been studied and modelled by Deloche and Seron (1982a, 1982b, 1987). Paths C and C′ allow access to the quantity code from arabic and verbal notation. They work by approximating the input numeral (e.g., extracting its highest power of ten) and activating the corresponding portion of the number line. Conversely, paths D and D′ retrieve approximately appropriate number names for a given quantity code. They presumably function by categorization of the number line continuum into segments of different lengths, each being assigned a specific arabic or verbal label (Dehaene & Mehler, 1992).

Two clarifications are in order. First, the translation paths to and from the magnitude representation are postulated to work only approximately and without any syntactic sophistication. For instance, path D′ (magnitude-to-verbal translation) does not contain a replication of the syntactic rules for composing any well-formed verbal numeral. Rather, it rigidly associates a "round" number name (e.g., "two hundred") to a given quantity by consulting its limited lexicon. For a more precise labelling of quantities (e.g., "two hundred and twelve"), verbal counting is required, which is a sophisticated coordination of the verbal number representation with pointing or other object tagging procedures. The second point is that the coexistence of routes C/D and routes C′/D′ is left open in the model. It is an empirical question whether direct transcoding to and from the magnitude representation is possible for both arabic and verbal notations, or whether there is a privileged notation, say arabic, for accessing the magnitude code. In the latter case, the magnitude of verbal numerals should not be accessible without an intermediate translation into arabic notation, and this should have measurable effects on reaction time.

Premise 2: each numerical procedure is tied to a specific input and output code. Each number-related task, however complex, can be decomposed into a sequence of component processes, each requiring a specific numerical format for input. The format in which numbers are manipulated must be assessed separately for each subcomponent of a task. This premise is in radical disagreement with McCloskey's

[13]The analytical arabic-to-verbal route, which can translate any arabic numeral to verbal notation using a complex syntactic algorithm, is perhaps supplemented by a more direct lexical route (McCloskey, this issue) storing frequently used arabic-to-verbal correspondences (Dehaene & Mehler, 1992). We have recently studied a patient whose performance suggests impaired analytical processing but spared lexical access in arabic numeral reading (Dehaene & Cohen, unpublished data). Patient AND was presented with 49 arabic numerals that were considered likely to be lexicalized (L numerals; e.g., 0–19, decades 20–90, famous dates, brands of cars . . .), and 48 other numerals (NL numerals; e.g., 21, 109, 710). He read 67.4% (33/49) of L numerals and only 16.7% (8/48) of NL numerals (χ^2 (1 d.f.) = 25.5, $p < .0001$). In our list, L numerals generally had a simpler structure than NL numerals. However, AND proved able to read relatively complex numerals in the L set (e.g., 75, 250, 1789), but not relatively simple numerals in the NL set (e.g., 22, 130, 801). With some L numerals that he was not able to read, AND made semantic paraphasias indicating that lexical access had occurred (e.g., he knew that 205 and 305 were brands of cars, or 421 the name of a game).

(this issue) hypothesis of a unique amodal representation. It is more in line with Campbell and Clark's (1988, 1992) encoding complex model, but not with their view that "the strength of specific code–function combinations may depend on an individual's idiosyncratic learning history, culture-specific strategies, and other factors" (p. 209; see also Gonzales & Kolers, 1982). The present model is deeply constrained by (1) limiting the mental codes for numbers to three, and (2) assigning each process to only one prespecified code.

On the basis of the preceding sections, several code–function assignments can be proposed (Figure 5). In section III evidence was presented suggesting that in *numerical comparison* the arabic input is transformed into an analogue magnitude code before the comparison can be performed (Moyer & Landauer, 1967; Dehaene, 1989; Dehaene et al., 1990). In section I we have seen that *multi-digit operations* seem to involve the mental manipulation of a spatial image of the operation in arabic notation (e.g., Ashcraft & Stazyk, 1981; see Dahmen, Hartje, Büssing, & Sturm, 1982; Hécaen, Angelergues, & Houillier, 1961). Conversely there is (modest) evidence suggesting that *addition and multiplication tables* are stored as verbal associations, and that bilingual subjects access them in the language that they acquired first (Gonzales & Kolers, 1987; see also Marsk & Maki, 1976). In Figure 5, I have therefore tentatively linked multi-digit operations with the visual arabic number form, and arithmetical fact retrieval with the auditory verbal word frame. A strong ensuing prediction is that subjects must switch mentally between the two notations in the course of complex calculations. Such translation operations should introduce a measurable cost in RT.

Finally, access to *parity information* is postulated to depend exclusively on the arabic code. The arguments that lead to this conclusion illustrate how empirical data can be used to constrain the triple-code model. Dehaene et al. (1991) measured the time to judge if a number was odd or even with a variety of input notations (arabic notation, normal or mirror-image verbal notation, and even the East arabic notation used in Iran). Similar response patterns were found in all cases, with for instance 0 and 1 being slow and the powers of 2 (2, 4, 8) being fast. This suggested that numerals are converted to a common format before access to parity information.

Is this common format arabic notation, verbal notation, or an abstract amodal representation? This was assessed by comparing parity judgment with two-digit numerals in arabic versus verbal notation. A congruity effect was disclosed with arabic numerals: responses were faster when both digits of the numeral shared the same parity (e.g., 24, 17) than when their parities differed (e.g., 14, 27; see Armstrong, Gleitman, & Gleitman, 1983). We interpreted this result as a kind of Stroop effect: in teens, for instance, the leftmost digit "1", because it is odd, facilitates the correct "odd" response to "17" and inhibits the correct "even" response to "14". However, the same result was obtained with verbal notation: relative to numerals "zero" to "nine", numerals "ten" to "nineteen" showed a facilitation for odd responses and an inhibition for even responses. Naturally the surface form of French teens numerals does not contain an element which, like the leftmost "1" in arabic notation, might bias the subject towards the "odd" response. On the contrary, in the French numerals "dix-sept" (17), "dix-huit" (18) and "dix-neuf" (19), the presence of the component word "dix" (10) might have been expected to facilitate the even response; actually the odd response was facilitated. This observation strongly suggests that verbal numerals were first translated internally into an arabic code on which parity was then assessed. A cost compatible with such an internal translation was indeed observed in RTs.[14]

In section III, evidence from the same parity judgment task also suggested an automatic activation of a magnitude code from arabic notation (path C in Figure 5). The observation of such activations of multiple codes in a task as simple as parity judgment argues for the heuristic value of the triple-code model.

GENERAL CONCLUSION

Modularity is the fundamental concept which emerges from the present review. According to the proposed model, the ideal of a unique "number concept", which motivated Piaget's (1952) or Frege's (1950) reflections, must give way to a fractionated set of numerical abilities, among which faculties such as quantification, number transcoding, calculation or approximation may be isolated. The triple-code model clusters these abilities into three groups according to the format in which numbers are manipulated. First there are those abilities, like verbal counting or arithmetical fact retrieval, that are parasitic on the general spoken or written language-processing system, and that use verbal numerical notation. They tax processes that are not particularly tailored for numbers. Addition and multiplication tables are just part of a learned lexicon of verbal associations, and the counting sequence is not different from other automatic series like the alphabet or month names. In such domains, concurrent breakdown of linguistic and numerical abilities is predicted in brain-lesioned patients (e.g., Dehaene & Cohen, 1991; Deloche & Seron, 1984).

Second, there are abilities like multi-digit calculation or parity judgment which require the mastery of a dedicated positional notation system, for instance, arabic notation. The invention of this notation and of the accompanying arithmetical algorithms is clearly contingent on linguistic competence and literacy, and it is therefore specifically human. Nevertheless, the arabic subsystem in its present form is entirely dedicated to numerical material, and it may therefore be conceptualized as separate. Support for this separation can be found in the

[14]McCloskey's model might still accommodate this result because its postulated semantic representation is similar to arabic notation in having separate slots for each power of ten. For discussion see Dehaene et al. (1991).

neuropsychological dissociation of acalculia from aphasia (Guttmann, 1937; Henschen, 1919) and in the existence of processing disorders for words or letters but not for arabic numerals (e.g., Anderson, Damasio, & Damasio, 1990).

Finally, the triple-code model considers the abilities to compare and to approximate numerical quantities as a third separate cluster. These abilities are present in animals, emerge in infants before the acquisition of language, and are therefore assumed to constitute a distinct preverbal system of arithmetic reasoning (Gallistel & Gelman, this issue). In human adults, the magnitude code plays a central role in understanding the quantity that a numeral represents and in checking the meaningfulness of calculations. It is the main, and perhaps the only, "semantic" representation of numbers. Processing numbers approximately via the analogue route may be the only numerical ability left in aphasic or acalculic patients who fail completely at manipulating number symbols (Dehaene & Cohen, 1991).

Adult human numerical cognition can therefore be viewed as a layered modular architecture, the preverbal representation of approximate numerical magnitudes supporting the progressive emergence of language-dependent abilities such as verbal counting, number transcoding, and symbolic calculation. Although the current research trend, reviewed in this issue, is to explore the deeper layers, or the "primitives" of numerical cognition, the alternative direction is likely to flourish in the coming years. How do we acquire and represent higher-order arithmetical concepts such as parity, divisibility or primality? By what psychological processes can the class of numbers be extended from integers to negative numbers, fractions (Gallistel & Gelman, this issue), real or even complex numbers? These largely unexplored questions may soon move from the realm of philosophy or epistemology to that of cognitive psychology (Kitcher, 1984).

References

Allik, J., & Tuulmets, T. (1991). Occupancy model of perceived numerosity. *Perception and Psychophysics*, *49*, 303–314.

Anderson, S.W., Damasio, A.R., & Damasio, H. (1990). Troubled letters but not numbers: Domain specific cognitive impairments following focal damage in frontal cortex. *Brain*, *113*, 749–766.

Antell, S.E., & Keating, D.P. (1983). Perception of numerical invariance in neonates. *Child Development*, *54*, 695–701.

Armstrong, S.L., Gleitman, L.R., & Gleitman, H. (1983). What some concepts might not be. *Cognitition*, *13*, 263–308.

Ashcraft, M.H. (1982). The development of mental arithmetic: A chronometric approach. *Developmental review*, *2*, 213–236.

Ashcraft, M.H. (1987). Children's knowledge of simple arithmetic: A developmental model and simulation. In J. Bisanz, C.J. Brainerd, & R. Kail (Eds.), *Formal methods in developmental psychology: Progress in cognitive developmental research* (pp. 302–338). New York: Springer.

Ashcraft, M.H., & Battaglia, J. (1978). Cognitive arithmetics: Evidence for retrieval and decision processes in mental addition. *Journal of Experimental Psychology: Human Learning and Memory*, *4*, 527–538.

Ashcraft, M.H., & Stazyk, E.H. (1981). Mental addition: A test of three verification models. *Memory and Cognition*, *9*, 185–196.
Assal, G., & Jacot-Descombes, C. (1984). Intuition arithmétique chez un acalculique. *Revue Neurologique*, *140*, 374–375.
Atkinson, J., Campbell, F.W., & Francis, M.R. (1976). The magic number 4 ± 0: A new look at visual numerosity judgments. *Perception*, *5*, 327–334.
Atkinson, J., Francis, M.R., & Campbell, F.W. (1976). The dependence of the visual numerosity limit on orientation, colour, and grouping in the stimulus. *Perception*, *5*, 335–342.
Attneave, F. (1962). Perception and related areas. In S. Koch (Ed.), *Psychology: A study of a science* (vol. 4). New York: McGraw-Hill.
Averbach, E. (1963). The span of apprehension as a function of exposure duration. *Journal of Verbal Learning and Verbal Behavior*, *2*, 60–64.
Baird, J.C., & Noma, E. (1975). Psychophysical study of numbers. I: Generation of numerical response. *Psychological Research*, *37*, 281–297.
Banks, W.P., & Coleman, M.J. (1981). Two subjective scales of number. *Perception and Psychophysics*, *29*, 95–105.
Banks, W.P., Fujii, M., & Kayra-Stuart, F. (1976). Semantic congruity effects in comparative judgments of magnitudes of digits. *Journal of Experimental Psychology: Human Perception and Performance*, *2*, 435–447.
Banks, W.P., & Hill, D.K. (1974). The apparent magnitude of number scaled by random production. *Journal of Experimental Psychology* (Monograph Series), *102*, 353–376.
Barbizet, J., Bindefeld, N., Moaty, F., & Le Goff, P. (1967). Persistances de possibilités de calcul élémentaire au cours des aphasies massives. *Revue Neurologique*, *116*, 170–178.
Baroody, A.J. (1983). The development of procedural knowledge: An alternative explanation for chronometric trends of mental arithmetic. *Developmental Review*, *3*, 225–230.
Baroody, A.J. (1984a). A reexamination of mental arithmetic models and data: A reply to Ashcraft. *Developmental Review*, *4*, 148–156.
Baroody, A.J. (1984b). More precisely defining and measuring the order-irrelevance principle. *Journal of Experimental Child Psychology*, *38*, 33–41.
Baroody, A.J., & Ginsburg, H.P. (1986). The relationship between initial meaningful and mechanical knowledge of arithmetics. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 75–112). Hillsdale, NJ: Erlbaum.
Beckwith, M., & Restle, F. (1966). The process of enumeration. *Psychological Review*, *73*, 437–444.
Benson, D.F., & Denckla, M.B. (1969). Verbal paraphasia as a source of calculation disturbances. *Archives of Neurology*, *21*, 96–102.
Besner, D., & Coltheart, M. (1979). Ideographic and alphabetic processing in skilled reading of English. *Neuropsychologia*, *17*, 467–472.
Bevan, W., & Turner, E.D. (1964). Assimilation and contrast in the estimation of number. *Journal of Experimental Psychology*, *67*, 458–462.
Bijeljac-Babic, R., Bertoncini, J., & Mehler, J. (1991). How do four-day-old infants categorize multisyllabic utterances? Manuscript submitted for publication.
Binet, A. (1981). *Psychologie des grands calculateurs et joueurs d'échecs*. Paris: Slatkine (original edition 1894).
Birnbaum, M.H. (1974). Using contextual effects to derive psychophysical laws. *Perception and Psychophysics*, *15*, 89–96.
Briars, D.J., & Siegler, R.S. (1984). A featural analysis of preschooler's counting knowledge. *Developmental Psychology*, *20*, 607–618.
Buckley, P.B., & Gillman, C.B. (1974). Comparison of digits and dot patterns. *Journal of Experimental Psychology*, *103*, 1131–1136.
Campbell, J.I.D. (1987). Network interference and mental multiplication. *Journal of Experimental Psychology: Learning, Memory and Cognition*, *13*, 109–123.
Campbell, J.I.D., & Clark, J.M. (1988). An encoding-complex view of cognitive number processing: Comment on McCloskey, Sokol & Goodman (1986). *Journal of Experimental Psychology: General*, *117*, 204–214.
Campbell, J.I.D., & Clark, J.M. (1989). Time course of error priming in number-fact retrieval:

Evidence for excitatory and inhibitory mechanisms. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 15*, 920–929.
Campbell, J.I.D., & Clark, J.M. (1992). Integrated versus modular theories of number skills and acalculia. *Brain and Cognition*, in press.
Campbell, J.I.D., & Graham, D.J. (1985). Mental multiplication skill: Structure, process, and acquisition. *Canadian Journal of Psychology, 39*, 338–366.
Caramazza, A., & McCloskey, M. (1987). Dissociations of calculation processes. In G. Deloche & X. Seron (Eds.), *Mathematical disabilities: A cognitive neuropsychological perspective* (pp. 221–234). Hillsdale, NJ: Erlbaum.
Chi, M.T.H., & Klahr, D. (1975). Span and rate of apprehension in children and adults. *Journal of Experimental Child Psychology, 19*, 434–439.
Chomsky, N. (1980). *Rules and Representations*. Oxford: Basil Blackwell.
Cohen, L., & Dehaene, S. (1991). Neglect dyslexia for numbers? A case report. *Cognitive Neuropsychology, 8*, 39–58.
Cooper, R.G. (1984). Early number development: Discovering number space with addition and subtraction. In C. Sophian (Ed.), *Origins of cognitive skills* (pp. 157–192). Hillsdale, NJ: Erlbaum.
Cuneo, D.O. (1982). Children's judgments of numerical quantity: A new view of early quantification. *Cognitive Psychology, 14*, 13–44.
Dahmen, W., Hartje, W., Büssing, A., & Sturm, W. (1982). Disorders of calculation in aphasic patients: Spatial and verbal components. *Neuropsychologia, 20*, 145–153.
Dantzig, T. (1967). *Number: The language of science*. New York: The Free Press.
Davis, D.L., & Ashmead, D.H. (1991). Infant's numerical discrimination between sets of sequentially presented figures. Manuscript submitted for publication.
Davis, H., & Pérusse, R. (1988). Numerical competence in animals: Definitional issues, current evidence and a new research agenda. *Behavioral and Brain Sciences, 11*, 561–615.
Dehaene, S. (1989). The psychophysics of numerical comparison: A reexamination of apparently incompatible data. *Perception and Psychophysics, 45*, 557–566.
Dehaene, S., & Cohen, L. (1991). Two mental calculation systems: A case study of severe acalculia with preserved approximation. *Neuropsychologia, 29*, 1045–1074.
Dehaene, S., Bossini, S., & Giraux, P. (1991). The mental representation of parity and numerical magnitude. Manuscript submitted for publication.
Dehaene, S., Dupoux, E., & Mehler, J. (1990). Is numerical comparison digital? Analogical and symbolic effects in two-digit number comparison. *Journal of Experimental Psychology: Human Perception and Performance, 16*, 626–641.
Dehaene, S., & Mehler, J. (1992). Cross-linguistic regularities in the frequency of number words. *Cognition, 43*, 1–29.
Deloche, G., & Seron, X. (1982a). From one to 1: An analysis of a transcoding process by means of neuropsychological data. *Cognition, 12*, 119–149.
Deloche, G., & Seron, X. (1982b). From three to 3: A differential analysis of skills in transcoding quantities between patients with Broca's and Wernicke's aphasia. *Brain, 105*, 719–733.
Deloche, G., & Seron, X. (1984). Semantic errors reconsidered in the procedural light of the stack concepts. *Brain and Language, 21*, 59–71.
Deloche, G. & Seron, X. (1987). Numerical transcoding: A general production model. In G. Deloche & X. Seron (Eds.), *Mathematical disabilities: A cognitive neuropsychological perspective* (pp. 137–170). Hillsdale, NJ: Erlbaum.
Duncan, E.M., & McFarland, C.E. (1980). Isolating the effects of symbolic distance and semantic congruity in comparative judgments: an additive-factors analysis. *Memory and Cognition, 8*, 612–622.
Folk, C.L., Egeth, H., & Kwak, H.W. (1988). Subitizing: Direct apprehension or serial processing? *Perception and Psychophysics, 44*, 313–320.
Frege, G. (1950). *The foundation of arithmetic*. Oxford: Basil Blackwell.
Frith, C.D., & Frith, U. (1972). The solitaire illusion: An illusion of numerosity. *Perception and Psychophysics, 11*, 409–410.
Fuson, K.C. (1982). An analysis of the counting-on solution procedure in addition. In T.P. Carpenter,

J.M. Moser, & T.A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 67–81). Hillsdale, NJ: Erlbaum.

Fuson, K.C. (1988). *Children's counting and concepts of number*. New York: Springer.

Fuson, K.C., & Hall, J.W. (1983). The acquisition of early number word meanings. In H. Ginsburg (Ed.), *The development of children's mathematical thinking* (pp. 49–107). New York: Academic Press.

Fuson, K.C., Pergament, G.G., & Lyons, B.G. (1985). Collection terms and preschoolers' use of the cardinality rule. *Cognitive Psychology, 17*, 315–323.

Gallistel, C.R. (1990). *The organization of learning*. Cambridge, MA: Bradford Books/MIT Press.

Gallistel, C.R., & Gelman, R. (1991). Subitizing: The preverbal counting mechanism. In W. Kessen, A. Ortony, & F. Craik (Eds:), *Festschrift for George Mandler*. Hillsdale, NJ: Erlbaum.

Geary, D.C., Widaman, K.F., & Little, T.D. (1986). Cognitive addition and multiplication: evidence for a single memory network. *Memory and Cognition, 14*, 478–487.

Gelman, R., & Gallistel, C.R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Gelman, R., & Meck, E. (1983). Preschooler's counting: Principles before skill. *Cognition, 13*, 343–359.

Gelman, R., & Mech, E. (1986). The notion of principle: The case of counting. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 29–57). Hillsdale, NJ: Erlbaum.

Gelman, R., Meck, E., & Merkin, S. (1986). Young children's numerical competence. *Cognitive Development, 1*, 1–29.

Gelman, R., & Tucker, M.F. (1975). Further investigations of the young child's conception of number. *Child Development, 46*, 167–175.

Gibbon, J., & Church, R.M. (1981). Time left: Linear versus logarithmic subjective time. *Journal of Experimental Psychology: Animal Behavior Processes, 7*, 87–107.

Ginsburg, N. (1976). Effect of item arrangement on perceived numerosity: Randomness vs. regularity. *Perceptual and motor skills, 43*, 663–668.

Ginsburg, N. (1978). Perceived numerosity, item arrangement, and expectancy. *American Journal of Psychology, 91*, 267–273.

Gonzalez, E.G., & Kolers, P.A. (1982). Mental manipulation of arithmetic symbols. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 8*, 308–319.

Gonzalez, E.G., & Kolers, P.A. (1987). Notational constraints on mental operations. In G. Deloche & X. Seron (Eds.), *Mathematical disabilities: A cognitive neuropsychological perspective* (pp. 27–42). Hillsdale, NJ: Erlbaum.

Graham, D.J. (1987). An associative retrieval model of arithmetic memory: How children learn to multiply. In J.A. Sloboda & D. Rogers (Eds.), *Cognitive Processes in mathematics* (pp. 123–141). Oxford: Clarendon Press.

Greeno, J.G., Riley, M.S., & Gelman, R. (1984). Conceptual competence and children's counting. *Cognitive Psychology, 16*, 94–143.

Groen, G.J., & Parkman, J.M. (1972). A chronometric analysis of simple addition. *Psychological Review, 79*, 329–343.

Guitel, G. (1975). *Histoire comparée des numérations écrites*. Paris, Flammarion.

Guttmann, E. (1937). Congenital arithmetic disability and acalculia (Henschen). *British Journal of Medical Psychology, 16*, 16–35.

Hayes, J.R. (1973). On the function of visual imagery in elementary mathematics. In W. Chase (Ed.), *Visual information processing* (pp. 177–214). New York: Academic Press.

Hécaen, H., Angelergues, R., & Houillier, S. (1961). Les variétés cliniques des acalculies au cours des lésions rétro-rolandiques: Approche statistique du problème. *Revue Neurologique, 105*, 85–103.

Henmon, V.A.C. (1906). The time of perception as a measure of differences in sensation. *Archives of Philosophy, Psychology, and Scientific Method, 8*, 5–75.

Henschen, S.E. (1919). Uber Sprach-, Musik- un Rechenmechanismen und ihre Lokalisationen im Großhirn. *Zeitschrift für die desamte Neurologie und Psychiatrie, 52*, 273–298.

Hinrichs, J.V., Beric, J.L., & Mosell, M.K. (1982). Place information in multidigit number comparison. *Memory and Cognition, 10*, 487–495.

Hinrichs, J.V., Yurko, D.S., & Hu, J.M. (1981). Two-digit number comparison: Use of place information. *Journal of Experimental Psychology: Human Perception and Performance, 7*, 890–901.

Hurford, J.R. (1975). *The linguistic theory of numerals.* Cambridge, UK: Cambridge University Press.

Hurford, J.R. (1987). *Language and number.* Oxford: Basil Blackwell.

Ifrah, G. (1981). *Histoire universelle des chiffres.* Paris: Seghers.

Indow, T., & Ida, M. (1977). Scaling of dot numerosity. *Perception and Psychophysics, 22*, 265–276.

Jamieson, D.G., & Petrusic, W.M. (1975). Relational judgments with remembered stimuli. *Perception and Psychophysics, 18*, 373–378.

Kaufman, E.L., Lord, M.W., Reese, T.W., & Volkmann, J. (1949). The discrimination of visual number. *American Journal of Psychology, 62*, 498–525.

Kitcher, P. (1984). *The nature of mathematical knowledge.* Oxford: Oxford University Press.

Klahr, D. (1973). Quantification processes. In W.G. Chase (Ed.), *Visual information processing* (pp. 3–34). New York: Academic Press.

Klahr, D., & Wallace, J.G. (1973). The role of quantification operators in the development of conservation. *Cognitive Psychology, 4*, 301–327.

Kolers, P.A. (1968). Bilingualism and information processing. *Scientific American, 218*, 78–86.

Krueger, L.E. (1972). Perceived numerosity. *Perception and Psychophysics, 11*, 5–9.

Krueger, L.E. (1982). Single judgments of numerosity. *Perception and Psychophysics, 31*, 175–182.

Krueger, L.E. (1989). Reconciling Fechner and Stevens: Toward a unified psychophysical law. *Behavioral and Brain Sciences, 12*, 251–267.

Krueger, L.E., & Hallford, E.W. (1984). Why 2 + 2 = 5 looks so wrong: On the odd-even rule in sum verification. *Memory and Cognition, 12*, 171–180.

Laming, D. (1989). Experimental evidence for Fechner's and Stevens' laws (commentary). *Behavioral and Brain Sciences, 12*, 277–281.

LeFevre, J., Bisanz, J., & Mrkonjic, L. (1988). Cognitive arithmetic: Evidence for obligatory activation of arithmetic facts. *Memory and Cognition, 16*, 45–53.

Mandler, G., & Shebo, B.J. (1982). Subitizing: An analysis of its component processes. *Journal of Experimental Psychology: General, 111*, 1–22.

Marsh, L.G., & Makki, R.H. (1976). Efficiency of arithmetic operations in bilinguals as a function of language. *Memory and Cognition, 4*, 459–464.

Matsuzawa, T. (1985). Use of numbers by a chimpanzee. *Nature, 315*, 57–59.

McCloskey, M., Aliminosa, D., & Sokol, S.M. (1992). Facts, rules, and procedures in normal calculation: Evidence from multiple single-patient studies of impaired arithmetic fact retrieval. *Brain and Cognition*, in press.

McCloskey, M., & Caramazza, A. (1987). Cognitive mechanisms in normal and impaired number processing. In G. Deloche & X. Seron (Eds.), *Mathematical disabilities: A cognitive neuropsychological perspective* (pp. 201–219). Hillsdale, NJ: Erlbaum.

McCloskey, M., Sokol, S.M., & Goodman, R.A. (1986). Cognitive processes in verbal-number production: Inferences from the performance of brain-damaged subjects. *Journal of Experimental Psychology: General*, 115, 307–330.

McCloskey, M., Sokol, S.M., Goodman-Schulman, R.A., & Caramazza, A. (1990). Cognitive representations and processes in number production: Evidence from cases of acquired dyscalculia. In A. Caramazza (Ed.), *Advances in cognitive neuropsychology and neurolinguistics* (pp. 1–32). Hillsdale, NJ: Erlbaum.

McCulloch, W.S. (1965). *Embodiments of mind.* Cambridge, MA: MIT Press.

Mech, W.H., & Church, R.M. (1983). A mode control model of counting and timing processes. *Journal of Experimental Psychology: Animal Behavior Processes, 9*, 320–334.

Menninger, K. (1969). *Number words and number symbols.* Cambridge, MA: MIT Press.

Miller, G.A. (1956). The magical number seven plus or minus two: Some limits on our capacity for processing information. *Pyschological Review, 63*, 81–97.

Miller, K.F., & Paredes, D.R. (1990). Starting to add worse: Effects of learning to multiply on children's addition. *Cognition*, *37*, 213–242.
Miller, K., Perlmutter, M., & Keating, D. (1984). Cognitive arithmetics: Comparison of operations. *Journal of Experimental Psychology: Learning, Memory and Cognition*, *10*, 46–60.
Miller, K., & Stigler, J.W. (1987). Counting in Chinese: Cultural variation in a basic cognitive skill. *Cognitive Development*, *2*, 279–305.
Minturn, A.L., & Reese, T.W. (1951). The effect of differential reinforcement on the discrimination of visual number. *Journal of Psychology*, *31*, 201–231.
Mitchell, R.W., Yao, P., Sherman, P.T., & O'Regan, M. (1985). Discriminative responding of a dolphin (*Tursiops truncatus*) to differentially rewarded stimuli. *Journal of comparative psychology*, *99*, 218–225.
Moore, D., Benenson, J., Reznick, J.S., Peterson, M., & Kagan, J. (1987). Effect of auditory numerical information on infant's looking behavior: contradictory evidence. *Developmental Psychology*, *23*, 665–670.
Morin, R.E., DeRosa, D.V., & Stultz, V. (1967). Recognition memory and reaction time. *Acta Psychologica*, *27*, 298–305.
Morton, J. (1970). A functional model of memory. In D.A. Norman (Ed.), *Models of human memory*. New York: Academic Press.
Moyer, R.S., & Landauer, T.K. (1967). Time required for judgements of numerical inequality. *Nature*, *215*, 1519–1520.
Noel, M.P., & Seron, X. (1992). Arabic number reading deficit: A single case study. *Cognitive Neuropsychology*, in press.
Parkman, J.M. (1972). Temporal aspects of simple multiplication and comparison. *Journal of Experimental Psychology*, *95*, 437–444.
Parkman, J.M., & Groen, G.J. (1971). Temporal aspects of simple addition and comparison. *Journal of Experimental Psychology*, *89*, 335–342.
Piaget, J. (1952). *The child's conception of number*. New York: Norton.
Poltrock, S.E., & Schwartz, D.R. (1984). Comparative judgments of multidigit numbers. *Journal of Experimental Psychology: Learning, Memory and Cognition*, *10*, 32–45.
Power, R.J.D., & Longuet-Higgins, H.C. (1978). Learning to count: A computational model of language acquisition. *Proceedings of the Royal Society (London)*, *B 200*, 319–417.
Restle, F. (1970). Speed of adding and comparing numbers. *Journal of Experimental Psychology*, *83*, 274–278.
Rilling, M., & McDiarmid, C. (1965). Signal detection in fixed ratio schedules. *Science*, *148*, 526–527.
Rosner, R.S. (1965). The power law and subjective scales of number. *Perceptual and Motor Skills*, *21*, 42.
Rule, S.J. (1969). Equal discriminability scale of number. *Journal of Experimental Psychology*, *79*, 35–38.
Rumbaugh, D.M., Savage-Rumbaugh, S., & Hegel, M.T. (1987). Summation in the chimpanzee (*Pan troglodytes*). *Journal of Experimental Psychology: Animal Behavior Processes*, *13*, 107–115.
Sagi, D., & Julesz, B. (1985). "Where" and "What" in vision. *Science*, *228*, 1217–1219.
Schaeffer, B., Eggleston, V.H., & Scott, J.L. (1974). Number development in young children. *Cognitive Psychology*, *6*, 357–379.
Schneider, B., Parker, S., Ostrosky, D., Stein, D., & Kanow, G. (1974). A scale for the psychological magnitude of numbers. *Perception and Psychophysics*, *16*, 43–46.
Sekuler, R., & Mierkiewicz, D. (1977). Children's judgments of numerical inequality. *Child Development*, *48*, 630–633.
Seron, X., & Deloche, G. (1983). From 4 to four: A supplement to "from three to 3". *Brain*, *106*, 735–744.
Seron, X., & Deloche, G. (1987). The production of counting sequences by aphasics: A matter of lexical processing? In G. Geloche & X. Seron (Eds.), *Mathematical disabilities: A cognitive neurpsychological perspective* (pp. 171–200). Hillsdale, NJ: Erlbaum.
Shepard, R.N., & Chipman, S. (1970). Second-order isomorphism of internal representations: Shapes of states. *Cognitive Psychology*, *1*, 1–17.

Shepard, R.N., Kilpatrick, D.W., & Cunningham, J.P. (1975). The internal representation of numbers. *Cognitive Psychology*, *7*, 82–138.

Shipley, E.F., & Shepperson, B. (1990). Countable entities: Developmental changes. *Cognition*, *34*, 109–136.

Siegler, R.S. (1987a). The perils of averaging data over strategies: An example from children's addition. *Journal of Experimental Psychology: General*, *116*, 250–264.

Siegler, R.S. (1987b). Strategy choices in subtraction. In J.A. Sloboda & D. Rogers (Eds.), *Cognitive processes in mathematics* (pp. 81–106). Oxford: Clarendon Press.

Siegler, R.S. (1988). Strategy choice procedures and the development of multiplication skill. *Journal of Experimental Psychology: General*, *3*, 258–275.

Siegler, R.S., & Robinson, M. (1982). The development of numerical understandings. In H.W. Reese & L.P. Lipsitt (Eds.), *Advances in child development and behavior* (Vol. 16, pp. 242–307). New York: Academic Press.

Siegler, R.S., & Shrager, J. (1984). Strategy choices in addition: How do children know what to do? in C. Sophian (Ed.), *Origins of cognitive skills* (pp. 229–293). Hillsdale, NJ: Erlbaum.

Song, M.J., & Ginsburg, H.P. (1988). The effect of the Korean number system on young children's counting: A natural experiment in numerical bilingualism. *International Journal of Psychology*, *23*, 319–332.

Sophian, C., & Adams, N. (1987). Infant's understanding of numerical transformations. *British Journal of Developmental Psychology*, *5*, 257–264.

Starkey, P., & Cooper, R.G., Jr. (1980). Perception of numbers by human infants. *Science*, *210*, 1033–1035.

Starkey, P., Spelke, E.S., & Gelman, R. (1983). Detection of intermodal numerical correspondences by human infants. *Science*, *222*, 179–181.

Starkey, P., Spelke, E.S., & Gelman, R. (1990). Numerical abstraction by human infants. *Cognition*, *36*, 97–127.

Stazyk, E.H., Ashcraft, M.H., & Hamann, M.S. (1982). A network approach to mental multiplication. *Journal of Experimental Psychology: Learning, Memory and Cognition*, *8*, 320–335.

Strauss, M.S., & Curtis, L.E. (1981). Infant perception of numerosity. *Child Development*, *52*, 1146–1152.

Strauss, M.S., & Curtis, L.E. (1984). Development of numerical concepts in infancy. In C. Sophian (Ed.), *Origins of cognitive skills* (pp. 131–156). Hillsdale, NJ: Erlbaum.

Svenson, O. (1975). Analysis of time required by children for simple additions. *Acta Psychologica*, *39*, 289–302.

Takahashi, A., & Green, D. (1983). Numerical judgments with Kanji and Kana. *Neuropsychologia*, *21*, 259–263.

Taves, E.H. (1941). Two mechanisms for the perception of visual numerousness. *Archives of Psychology*, *37*, no. 265.

Teghtsoonian, R., & Teghtsoonian, M. (1989). Unified psychophysics: Wouldn't it be loverly . . . (commentary). *Behavioral and Brain Sciences*, *12*, 292.

Thomas, R.K., & Chase, L. (1980). Relative numerousness judgments by squirrel monkeys. *Bulletin of the Psychonomic Society*, *16*, 79–82.

Timmers, L., & Claeys, W. (1990). The generality of mental addition models: Simple and complex addition in a decision-production task. *Memory and Cognition*, *18*, 310–320.

Treiber, F., & Wilcox, S. (1984). Discrimination of number by infants. *Infant Behavior and Development*, *7*, 93–100.

Trick, L.M., & Pylyshyn, Z.W. (1988). When subitizing fails: The importance of preattentive item indexing for subitizing. University of Western Ontario COGMEM #35.

Trick, L.M., & Pylyshyn, Z.W. (1989). Subitizing and the FINST spatial index model. University of Western Ontario COGMEM #44.

Trick, L.M., & Pylyshyn, Z.W. (1991). A theory of enumeration that grows out of a general theory of vision: Subitizing, counting and FINSTs. University of Western Ontario COGMEM #57.

Ullman, S. (1984). Visual routines. *Cognition*, *18*, 97–159.

van Lehn, K. (1986). Arithmetic procedures are induced from examples. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 133–179). Hillsdale, NJ: Erlbaum.

van Lehn, K. (1990). *Mind bugs: The origin of procedural misconceptions*. Cambridge, MA: MIT Press.

van Loosbroek, E., & Smitsman, A.W. (1990). Visual perception of numerosity in infancy. *Developmental Psychology*, *26*, 916–922.

van Oeffelen, M.P., & Vos, P.G. (1982). A probabilistic model for the discrimination of visual number. *Perception & Psychophysics*, *32*, 163–170.

Vos, P.G., van Oeffelen, M.P., Tibosch, H.J., & Allik, J. (1988). Interactions between area and numerosity. *Psychological Research*, *50*, 148–154.

Warrington, E.K. (1982). The fractionation of arithmetical skills: A single case study. *Quarterly Journal of Experimental Psychology*, *34A*, 31–51.

Washburn, D.A., & Rumbaugh, D.M. (1991). Ordinal judgments of numerical symbols by macaques (*Macaca mulatta*). *Psychological Science*, *2*, 190–193.

Weddel, R.A., & Davidoff, J.B. (1991). A dyscalculic patient with selectively impaired processing of the numbers 7, 9 and 0. *Brain and Cognition*, *17*, 240–271.

Widaman, K.F., Geary, D.C., Cormier, P., & Little, T.D. (1989). A componential model for mental addition. *Journal of Experimental Psychology: Learning, Memory and Cognition*, *15*, 898–919.

Winkelman, H.J., & Schmidt, J. (1974). Associative confusions in mental arithmetic. *Journal of Experimental Psychology*, *102*, 734–736.

Woodruff, G., & Premack, D. (1981). Primative (sic) mathematical concepts in the chimpanzee: Proportionality and numerosity. *Nature*, 293, 568–570.

Wynn, K. (1990). Children's understanding of counting. *Cognition*, *36*, 155–193.

Zbrodoff, N.J., & Logan, G.D. (1986). On the autonomy of mental processes: A case of study of arithmetic. *Journal of Experimental Psychology: General*, *115*, 118–130.

Zbrodoff, N.J., & Logan, G.D. (1990). On the relation between production and verification tasks in the psychology of simple arithmetic. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *16*, 83–97.